Why I Love Mathematics

Where to begin? I have loved numbers, and the relationships among numbers, ever since I first learned to count. I was about three years old, sitting on the floor of our quonset hut in Alaska. My mother was ironing clothes. I had a small plastic lid from an old coffee can in my hand, and I already knew how to count to ten.

I tossed the lid so it bounced off the ceiling. "One," I said. Again. "Two." And so I went on. "Three. four, five. six, seven, eight, nine, ten," I said, tossing the lid each time.

I looked up at my mother. "Eleven", she said, helpfully. "Twelve, thirteen, fourteen."

"Fiveteen?" I ventured. "Fifteen," she said. "Sixteen?" She nodded. "Seventeen, eighteen, nineteen," said I.

Another brief pause. "Twenty, twenty-one," she prompted. "Twenty-two, twenty-three, twenty-four, twenty-five, ..."

"Threety!" I said, triumphantly. "Not quite. Thirty." "Thirty-one, thirty-two, thirty-three ..." She nodded her approval. "Five-ty" became fifty, but everything else worked just right. I rattled on: "... ninety-seven, ninety-eight, ninety-nine." Again I looked up for assistance.

"One hundred, then one hundred and one, " she said. "You're giving me a headache. Go play outside." And she threw me outdoors, so she could continue her ironing in peace.

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I'm not sure why I love mathematics. Maybe it's just because I have a natural affinity for numbers. Or maybe it's because the innate patterns and rhythms of the integers appeal to my sense of that which is æsthetically beautiful. All I know for certain is that logic – mathematical logic – seems to me the pinnacle of human achievement. And that's why I have decided to write a series of discursive essays describing numbers and the way I understand them, in hopes that someone somewhere who shares my love for numbers might eventually share in the joy of discovery. That, plus I have a lot of spare time right now. And numbers are beautiful!

Before we start to explore the world of numbers, we should talk about logic. For many years "formal logic" and "mathematics" were regarded as distinct, though related, disciplines. The more modern perspective – beginning with George Boole, who flourished during the middle years of the nineteenth century, in Ireland – is that logic is a part of mathematics, root and branch. I don't intend to get real formal, with lots of scary symbols, right away. Let's just start by thinking about the techniques people use to distinguish truth from falsehood.

The first thing we need to get straight is the difference between a definite statement that is verifiable and a vague or ambiguous statement which, in the final analysis, doesn't really mean anything. Right away we're up against a problem that the proponents of formal logic identified just about a hundred years ago: "natural" languages are inherently ambiguous. A program of "formalization" ensued: an entire generation of mathematicians devoted their lives to the impossible dream of eliminating all ambiguity from mathematical language. Eventually they learned that their efforts were misguided. No matter how hard we try to eliminate ambiguities, in any system of formal logic that allows a structure as complex as the natural numbers $\{1, 2, 3, 4, 5, \cdots\}$ to exist, some statements can be formulated that cannot be proved to be either true or false. This result is known as Gödel's First Incompleteness Theorem. So in a certain sense the attempt to eliminate all ambiguities from mathematics was a failure. In a larger sense it was a huge success: Gödel's theorems demonstrate that there are limits to logic, which logic itself can recognize, but which can never be eliminated entirely.

For our present purposes, we can rely on your intuitive understanding of ambiguity. The statement "This man is a very industrious worker" is clear and unambiguous, as is the statement "This fellow is lazy." But the statement "Nobody can do a better job than he does" is ambiguous. Is he the very best worker on the planet? Or would the company not miss him at all if he just went away?

If you're old enough, you may remember Martin Gardner's *Mathematical Games* column, which ran in *Scientific American* for many years. One of the first math books I ever read was Gardner's *Mathematics, Magic, and Mystery.* I think that's where I ran across an old riddle that illustrates the ambiguity of natural languages almost perfectly. **Q.** Which is better, complete happiness in life, or a ham sandwich? **A.** A ham sandwich. Nothing is better than complete happiness in life, and a ham sandwich is better than nothing!

It's also possible for a statement to be self-contradictory. I'm sure you can construct an example or two without my help.

Why do I mention this? Well, mathematicians aren't generally interested in statements that are ambiguous, vague, or self-contradictory. We want to make definite statements that we know to be true. Our general procedure is to begin with a set of postulates, or axioms, whose truth is taken for granted. In traditional mathematics (I'm thinking of Euclidean Geometry, in particular) these postulates were usually regarded as self-evident truths, as in "There is one and only one straight line between two points." More recently – let's say since the time of Gauss, who was born in 1777 – mathematical postulates are regarded as necessary for the development of a particular branch or field of inquiry, but not literally "self-evident". The classic example of this distinction arose from the attempts to "prove" Euclid's parallel postulate (through any point not lying on a given line, there exists a unique line that is parallel to the given line). After centuries of struggle, mathematicians finally realized that (a) the parallel postulate is independent of the other four rules Euclid adopted, and (b) it is possible to construct logically consistent geometries in which there are either no parallel lines at all (known as spherical, Riemannian, or elliptic geometry), or in which any given line has infinitely many parallels (as described by Lobachevsky, Bolyai, and Gauss).

Anyway, once we have agreed upon a set of postulates, and have defined some objects related to those postulates in one way or another, we proceed to use the laws of deductive logic to derive *theorems*, which are statements that are logically equivalent to the postulates themselves. This is the method that Euclid employed in his classic treatise on geometry, *The Elements.* This basic procedure has remained the same for nearly 2,500 years. Clearly, Euclid had a good idea. If we start with some propositions that we know are true (or that we agree to accept as true), and then reason logically to prove additional statements, all the statements so constructed will also be true (so long as there were no contradictions inherent in our initial postulates, and we don't make any mistakes in our reasoning).

I should insert a brief word here on the difference between *inductive logic* and *deductive logic*. Aristotle taught that there are two kinds of reasoning: from the particular to the general, and from the general to the particular. Aristotle's first procedure – reasoning from a mass of empirical observations in hopes of ascertaining some general principle, or rule – is known as induction. His second procedure, the procedure that Euclid popularized for geometry, is known as deduction. Propositions (or *theories*) based at least partially on induction are somewhat dubious, because no matter how many empirical observations we make, we cannot be certain that our next observation will not contradict a conclusion we have already drawn. Mathematicians do not rely on inductive logic (except to the extent that the postulates initially chosen are rooted in their experience of physical reality). Once a list of postulates has been compiled, the mathematician uses only the laws of deductive logic to build his superstructure of theorems. Empirical evidence about mathematical objects is useful, because it may lead the mathematician to put forward new conjectures, which eventually become theorems, if proven. But empirical evidence alone is never sufficient to constitute a proof.

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The Laws of Logic and Methods of Proof

I have already spoken of the "laws of logic," so you may very well be wondering what those laws are. Historically, logicians have spoken of three basic logical laws: The law of identity, the law of non-contradiction, and the law of the excluded middle. These laws are most conveniently understood symbolically. So let us suppose that A is a definite statement that may be either True or False, and let us define a negation operator $\neg$ that reverses the truth value of any statement A: if A is True, then $\neg A$ is False, and vice-versa. With these conventions, the three traditional laws of logic may be stated succinctly.

Identity: A is A ("Whatever is, is.")

Non-contradiction: $\neg(A$ and $\neg A)$ ("Nothing is both True and False.")

Excluded Middle: A or $\neg A$ ("Everything either is, or it is not.")

To these three laws we may append the principle of *modus ponens*, another name for the classic syllogism "P implies Q, and P is true, therefore Q follows" and the rule of the contrapositive "P implies Q, and Q is false, therefore P is also false". That's just about it, so far as the laws of logic go. (Not every mathematician would agree with me on this point. For example, Bertrand Russell and Alfred North Whitehead, in *Principia Mathematica*, claimed that there are eight basic laws of logic. I regard their position as pettifogging excess.) Proofs are a little more complicated. The following list is illustrative, and not exhaustive. (In the examples given below, I have employed some common mathematical notions – like *prime* and *composite* natural numbers – that have not yet been defined. If that's confusing, please read the next chapter, then come back and read through the examples once again. If you're still confused, please send me a message, and I'll do my best to explain the unfamiliar terminology.)

A **constructive proof** proceeds by exhibiting a mathematical object that has the desired properties. For instance, one of the number theory books I studied years ago asked me to prove that the cube of any natural number can be expressed as the difference between two perfect squares. I gave a constructive proof. First we observe that $n^3 = n * n^2$. If n is an odd number, n^2 is also an odd number. If n is even, n^2 is even. Therefore both the sum $n^2 + n$ and difference $n^2 - n$ are always even numbers, because the sum and difference of two even numbers are both even, as are the sum and difference of two odd numbers. Now the pair of simultaneous equations $a + b = n^2$ and $a - b = n$ can be solved: $a = \frac{1}{2}(n^2 + n)$ and $b = \frac{1}{2}(n^2 - n)$. Because both $n^2 + n$ and $n^2 - n$ are even, these solutions are natural numbers, and $a^2 - b^2 = (a + b)(a - b) = n^2 n = n^3$ is the desired solution.

An **existence proof** is fundamentally different. It merely shows that a particular mathematical object must exist, without offering a specific concrete instance of the object under consideration. Here is a very cute example: Prove that there is at least one irrational number p and one irrational number q such that p^q is a rational number.

We observe first that $\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \sqrt{2}^2 = 2$. If $x = \sqrt{2}^{\sqrt{2}}$ is irrational, choose $p = x$, $q = \sqrt{2}$ and the theorem has been proved. If x is a rational number, then chooose $p = q = \sqrt{2}$ to prove the theorem. (This example is *quasi-constructive*, because it involves just two possibilities, one of which must be the object we would like to construct. Many existence proofs are much more abstract, leaving us without even the germ of an idea how the object we would like to exhibit can be constructed. More on that later.)

A **proof by implication** or simple "if, then" proof is probably the most common sort of proof in mathematics, generally speaking. It is in the form of the *modus ponens*: A implies B, and A is true, therefore B must be true. The most common statement of the famous Pythagorean Theorem is in this form: If the angle between the two shorter sides of length a and b in a triangle with sides of length a, b, c is a right angle, then $a^2 + b^2 = c^2$. When we can prove that A implies B, mathematicians often say that the truth of A is a *sufficient* condition to establish the truth of B.

The **proof of logical equivalence**, or "if and only if" style is similar to the "if, then" proof, except that the implication goes both ways. That is, if we can show that A implies B, and also that B implies A, then the two statements A and B are said to be *logically equivalent*. The aforementioned Pythagorean Theorem is a good example, because it can also be shown that the equation $a^2 + b^2 = c^2$ is true *only if* the angle between the two shorter sides a, b of the triangle with sides of length a, b, c is a right angle. When the statement A is logically equivalent to B, mathematicians sometimes say that the truth of A is both *necessary and sufficient* to establish the truth of B. (Of course, B is also a necessary and sufficient condition for A, because the two statements are *equivalent*.)

The **proof by contradiction**, or *reductio ad absurdum*, is based squarely on the law of the excluded middle. We begin by assuming that a particular statement A is true, then demonstrate that this assumption produces a contradiction – *i.e.*, it leads to a conclusion that is clearly false. We then conclude that A cannot possibly be true and hence, by the law of the excluded middle, it must be false.

Here is a classic argument by *reductio ad absurdum*, from Euclid's *Elements* (Book VII, Proposition 31). Prove that every composite natural number N has at least one prime factor. Assume the negation, that there is a composite natural number N that has no prime factors. Then there must be a smaller number N_1 that divides N evenly, because N is composite by hypothesis. Now N_1 cannot be prime, because N has no prime factors.

Therefore there is a still smaller composite number N_2 that divides N_1 evenly. Proceeding in this fashion we can construct an infinite sequence of composite natural numbers $N > N_1 > N_2 > \cdots$. But this is impossible because every natural number is greater than zero; there are only finitely many natural numbers less than any particular N. We conclude that there cannot be a composite natural number N that has no prime factors, or, equivalently, that every composite natural number N has at least one prime factor.

The **proof by mathematical induction** is not an inductive proof, despite its name. It is in fact an infinitely long chain of "if, then" proofs all rolled up into a single argument. Given a statement about the natural numbers $\{1, 2, 3, 4, 5, \cdots\}$ we show first that if a statement is true for a particular number n, it is necessarily true for $n+1$. If we can then also show that the statement is true for the value 1, we conclude that the statement is true for every natural number. It's sort of like a long train of dominoes standing on end. Once the first domino falls over, all the rest must also fall. (The generalizations are pretty obvious – for example, if something is only true when $n > 3$, we start the "induction" with $A(4) \Longrightarrow A(5) \Longrightarrow A(6) \Longrightarrow \cdots$. And when we start dealing with the continuum, we may occasionally resort to "transfinte" induction. But all of these cases ultimately look like a long row of dominoes, each one of which must fall once the first one topples over.)

Here's a simple proof by mathematical induction. The **triangular numbers** T_n are defined to be the sum of the first n natural numbers. So $T_1 = 1, T_2 = 1+2 = 3, T_3 = 1+2+3 = 6$, etc. Prove that $T_n = \frac{1}{2}n(n+1)$. Suppose that the given formula holds for $n-1$; that is, assume that $T_{n-1} = \frac{1}{2}(n-1)n$. By simple algebra we see that

$$T_n = T_{n-1} + n \text{ (by definition) } = \frac{(n-1)n}{2} + n = \frac{n^2 - n + 2n}{2} = \frac{n^2 + n}{2} = \tfrac{1}{2}n(n+1)$$

and, since $T_1 = 1 = \frac{1}{2} \cdot 1 \cdot (1+1) = \frac{1}{2} \cdot 2$, the induction is complete.

OK, one more method of proof, and then we can move on to the next topic, the construction of systems of enumeration, and number systems in general. The **proof by infinite descent** was made famous by Pierre de Fermat, who used it to prove many theorems about natural numbers (although few of those proofs were actually recorded and preserved). The proof given above to illustrate the *reductio ad absurdum* is also an example of infinite descent: the argument begins by assuming that some natural number has a particular property, then argues that some smaller number must have the same property, and ends by showing either that the process never ends (which is impossible, for any decreasing sequence of natural numbers must be of finite extent) or that it leads to one smallest natural number in the chain that does not in fact have the desired property, so that the original supposition must be false.

Here is an example of a strictly facetious "proof" by infinite descent, which circulates among mathematicians as a somewhat nerdy joke. **Theorem:** Every natural number is unusual. **Proof:** Assume that some natural number N exists which is not unusual. Since the set of natural numbers is bounded below, the set Ω of all natural numbers that are not unusual has a smallest element. Call that smallest element of Ω N_0. Now N_0 is the smallest natural number that is not unusual. But this implies that N_0 is in fact unusual, because it is the smallest number in Ω. Therefore the assumption is false, and the theorem is proved.

(Can you spot the logical fallacy in this purported "proof"? Think about it for a few seconds before you read on. Got it? The "proof" relies on the ambiguity of natural language.

We err when we talk about "unusual" natural numbers without defining exactly what an "unusual" number is, and then use it to mean more than one thing. Sort of like "nothing" in the joke about a ham sandwich and complete happiness in life.)

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Next: Systems of Enumeration and Number Systems