Leonhard Euler and the Infinitude of Primes

If you have ever seen a proof that there are infinitely many prime numbers, it was most likely Euclid's classic proof by *reductio ad absurdum*: Book IX, Proposition 20.

The number of primes is not finite. Suppose that we can list all the prime numbers, and suppose further that the list of all the primes is $\{p_0, p_1, p_2, \ldots, p_n\}$. Consider the number $N = p_0 \cdot p_1 \cdot p_2 \cdot \ldots \cdot p_n + 1$. Clearly N has at least one prime factor that is greater than unity. But N cannot be divided evenly by any one of the $\{p_i\}$, by construction. So there is at least one prime number not on the list with which we started. ★

Today I'd like to introduce you to another proof of the same conclusion that was concocted by Leonhard Euler, a great Swiss mathematician who was born in the year 1707 A.D. Just to put Euler in historical context, he was born not long before Sir Isaac Newton died, and he himself passed away about the time the United States of America won its independence from Britain. Most historians of mathematics rate Euler as one of the greatest mathematicians who has ever lived, right behind Archimedes, Newton, and Gauss (who was born just about five years before Euler died). Mathematical analysis, or calculus, was invented in the mid-seventeenth century (by Newton, and Leibniz), came of age during the eighteenth century (Euler), and reached its modern form during the nineteenth century (Gauss, Riemann, Cauchy, *et al.*), broadly speaking.

In my estimation, Euler is primarily memorable because of the way he was able to manipulate infinite sums, and infinite products. Before plunging into the "Euler Zeta Function" and Euler's very innovative proof that there are infinitely many primes, it's probably wise to review some now standard mathematical notation; in particular, the summation sign $\sum$, and the extended product sign $\prod$. These signs (capital Greek letter sigma, or Σ, for a sum, and capital Greek letter pi, or Π, for a product) were popularized by Euler and Gauss, respectively, and are defined as follows.

$$\sum_{i=1}^{n} x_i = S \tag{1}$$

is read as "the sum from $i = 1$ to $i = n$ of the summands x_i is equal to S", or, in symbols:

$$\sum_{i=1}^{n} x_i = x_1 + x_2 + x_3 + \cdots + x_n = S \tag{2}$$

The product symbol works the same way.

$$\prod_{i=1}^{n} x_i = P \tag{3}$$

is read as "the product from $i = 1$ to $i = n$ of the factors x_i is equal to P", or, in symbols:

$$\prod_{i=1}^{n} x_i = x_1 \cdot x_2 \cdot x_3 \cdot \ \ldots \cdot x_n = P \tag{4}$$

The Infinite Geometric Series

A sequence $\{1, x_1, x_2, x_3, \cdots\}$ is said to be in geometric progression if the terms are in constant ratio, that is, if $\{1, x_1, x_2, x_3, \cdots\} = \{r^0, r^1, r^2, r^3, \cdots\}$ for some fixed quantity r. To be clear, the first term in a geometric progression can be any number at all (except zero). But since the geometric progression $\{k, kr, kr^2, kr^3, \cdots\}$ is conveniently represented as $k \cdot \{1, r, r^2, r^3, \cdots\}$, we can put our examples of geometric progresssions in a standard form, and avoid a little bit of complexity in what follows.

Now if r is any given number, we can evaluate the sum of the first n terms in the geometric progression $\{1, r, r^2, r^3, \cdots\}$ via the following argument.

$$\begin{aligned} S &= 1 + r + r^2 + r^3 + \cdots + r^{n-1} \\ r \cdot S &= \quad\; r + r^2 + r^3 + \cdots + r^{n-1} + r^n \end{aligned}$$

Subtracting the second equation from the first one we obtain the general summation formula for a geometric series:

$$(1-r)S = 1 - r^n \quad \Longrightarrow \quad S = \frac{(1-r^n)}{(1-r)} \tag{5}$$

A moment's reflection should convince you that, if r is in the range $-1 < r < 1$, the term r^n in equation (5) above will tend to zero as n increases without bound, so that we may write

$$\sum_{i=0}^{\infty} r^i = \frac{1}{1-r} \quad \text{whenever} \quad |r| < 1. \tag{6}$$

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Euler's Zeta Function

You may have heard of Riemann's Zeta Function $\zeta(s)$ and the "Riemann Hypothesis", which states that the only complex numbers $s = x + iy$ for which $\zeta(s) = 0$ are the negative even integers $\{-2, -4, -6, \cdots\}$ and another denumerably infinite set of points lying along a vertical line in the Argand plane: $\{\frac{1}{2} + iy\}$. While Riemann definitely deserves the credit for extending the Zeta Function throughout the complex plane, and for discovering its intimate association with the distribution of prime numbers, he was following in the footsteps of Euler, who made the initial discoveries about some fundamental properties of the Zeta Function about a century before Riemann published his seminal paper on the subject. Let's retrace those early steps.

We begin by observing that if p is a prime number, the quantity $\frac{1}{p}$ lies between 0 and 1, or, in symbols, $|\frac{1}{p}| < 1$ for every prime number p. So we can apply equation (6) for the infinite sum of a geometric series to the reciprocal of each one of the primes in turn to obtain the equalities

$$\frac{1}{1-\frac{1}{2}} = 1 + \tfrac{1}{2} + \tfrac{1}{2^2} + \tfrac{1}{2^3} + \cdots$$
$$\frac{1}{1-\frac{1}{3}} = 1 + \tfrac{1}{3} + \tfrac{1}{3^2} + \tfrac{1}{3^3} + \cdots$$
$$\frac{1}{1-\frac{1}{5}} = 1 + \tfrac{1}{5} + \tfrac{1}{5^2} + \tfrac{1}{5^3} + \cdots$$
$$\frac{1}{1-\frac{1}{7}} = 1 + \tfrac{1}{7} + \tfrac{1}{7^2} + \tfrac{1}{7^3} + \cdots$$

and so forth. But now if we just apply the ordinary rules of algebra, and truncate each of the series appearing on the right above, we see an interesting pattern emerging:

$$(1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \cdots)(1 + \tfrac{1}{3} + \tfrac{1}{9} + \cdots)(1 + \tfrac{1}{5} + \cdots)(1 + \tfrac{1}{7} + \cdots)$$
$$= 1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{7} + \tfrac{1}{8} + \tfrac{1}{9} + \tfrac{1}{10} + \cdots$$

Euler saw in a flash that he could apply the Fundamental Theorem of Arithmetic (every natural number greater than one is either a prime number, or else it can be uniquely expressed as the product of smaller prime numbers) to write down the fundamental equation that underlies the famous Zeta Function:

$$\prod_{primes\ p} \frac{1}{1-\frac{1}{p}} = \prod_{primes\ p} \frac{p}{p-1} = \sum_{n=1}^{\infty} \frac{1}{n} \quad \text{or, equivalently,} \quad \prod_{primes\ p} \frac{1}{1-\frac{1}{p^s}} = \sum_{n=1}^{\infty} \frac{1}{n^s} \qquad (7)$$

where the product is to be taken over all the prime numbers p, and the exponent s, in Euler's formulation, is a positive rational number. (Riemann generalized this formula to include complex values of s, and that generalization was so powerful and far-reaching the we attribute the Zeta Function to Riemann, even though Euler was the first mathematician to express this relationship between a product involving only the prime numbers and a sum involving all the natural numbers.)

The infinitude of the prime numbers follows immediately from equation (7) above, because the sum on the right hand side (when $s = 1$) is known to be infinitely large, and the product would definitely be finite if there only finitely many prime numbers. (A short demonstration that the sum in (7) is divergent follows, in case you haven't learned that already.) ★

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On the Divergence of the Harmonic Series

There are many ways to show that the sum represented by the Harmonic Series $1 + \frac{1}{2} + \frac{1}{3} + \cdots$ is infinitely large. I think that this proof, first given by Pietro Mengoli in 1647, is particularly cute. We must first establish an auxiliary lemma:

For any natural number $a > 1$, $\frac{1}{a-1} + \frac{1}{a} + \frac{1}{a+1} > \frac{3}{a}$.

Proof: If $a > 1$, it is clear that $2a^3 > 2a^3 - 2a = 2a(a^2 - 1)$. If we divide this inequality by $a^2(a^2 - 1)$ we obtain

$$\frac{2a^3}{a^2(a^2-1)} > \frac{2a(a^2-1)}{a^2(a^2-1)} \quad \Longrightarrow \quad \frac{2a}{(a^2-1)} > \frac{2}{a}$$

But now it's easy to complete the argument:

$$\begin{aligned}\frac{1}{a-1} + \frac{1}{a} + \frac{1}{a+1} &= \frac{1}{a} + \left(\frac{1}{a-1} + \frac{1}{a+1}\right) \\ &= \frac{1}{a} + \frac{2a}{a^2-1} > \frac{1}{a} + \frac{2}{a} = \frac{3}{a} \quad \bigstar\end{aligned}$$

Having proved this preliminary result, we can easily show that the sum $S = \sum_{n=1}^{\infty} \frac{1}{n}$ is infinitely large:

$$\begin{aligned}S &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \cdots \\ &= 1 + \left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7}\right) + \left(\frac{1}{8} + \frac{1}{9} + \frac{1}{10}\right) + \cdots \\ &> 1 + \frac{3}{3} + \frac{3}{6} + \frac{3}{9} + \frac{3}{12} + \frac{3}{15} + \frac{3}{18} + \frac{3}{21} + \frac{3}{24} + \frac{3}{27} + \frac{3}{30} + \cdots \\ &= 1 + 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \cdots \\ &= 1 + 1 + \left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7}\right) + \left(\frac{1}{8} + \frac{1}{9} + \frac{1}{10}\right) + \cdots \\ &> 1 + 1 + \frac{3}{3} + \frac{3}{6} + \frac{3}{9} + \frac{3}{12} + \frac{3}{15} + \frac{3}{18} + \frac{3}{21} + \frac{3}{24} + \frac{3}{27} + \frac{3}{30} + \cdots\end{aligned}$$

whence it is clear that the sum S is unbounded. ★