Introduction to Continued Fractions

The typical undergraduate course in mathematical analysis begins with the notion of a limit, first illustrated by an infinite sequence that is bounded, and later by infinite series and infinite products. Eventually derivatives and anti-derivatives, or integrals, are also introduced, and the values associated with these infinite processes are rigorously defined by reference to the notion of a limit.

Unfortunately, most math students remain blissfully unaware of a more intuitively appealing procedure for expressing an irrational number as the limit of an infinite process. *Continued fractions* have a long and rich history commencing with Rafael Bombelli, who wrote about them in 1579. Leonhard Euler and Karl Friedrich Gauss, two of the most influential mathematicians of all time, devoted considerable time and effort to understanding continued fractions, and applied them to problems in analysis. Indeed, most of the modern computer algorithms for analytic functions – such as the logarithm function, the exponential function, and the trigonometric functions – are based on continued fractions.

What is a continued fraction? The easiest way to explain the idea is to derive a simple continued fraction algebraically. So let's start by solving the equation $x^2 = 2$. And let's do this by using only the elementary arithmetic operations of addition, subtraction, multiplication, and division.

We begin by transforming our simple quadratic equation using the ordinary rules of algebra.

$$x^2 = 2 \tag{1}$$

$$x^2 - 1 = 1 \tag{2}$$

$$(x-1)(x+1) = 1 \tag{3}$$

$$x - 1 = \frac{1}{1+x} \tag{4}$$

$$x = 1 + \frac{1}{1+x} \tag{5}$$

Now for the tricky part. Notice that the variable x occurs only once on the right hand side of the transformed equation. So let's substitute the entire right hand side of this equation, which represents x, in place of x itself.

$$x = 1 + \frac{1}{1+x} \tag{6}$$

$$x = 1 + \frac{1}{1 + \left(1 + \dfrac{1}{1+x}\right)} \tag{7}$$

$$x = 1 + \frac{1}{2 + \dfrac{1}{1+x}} \tag{8}$$

But if we can do this once we can do it again!

$$x = 1 + \cfrac{1}{2 + \cfrac{1}{1+x}} \tag{9}$$

$$x = 1 + \cfrac{1}{2 + \cfrac{1}{1 + \left(1 + \cfrac{1}{1+x}\right)}} \tag{10}$$

$$x = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{1+x}}} \tag{11}$$

And if we can do it twice, we can do it *ad infinitum*. Recognizing that the solution to our original equation ($x^2 = 2$) is the square root of 2, we have an infinitely long continued fraction representation of that famous irrational number.

$$\sqrt{19} = 4 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{8 + \ddots}}}}}} \tag{12}$$

Calculating the Successive Convergents to $\sqrt{2}$

We may obtain an infinite series of rational approximations to $\sqrt{2}$ by evaluating successively longer truncations of the infinite continued fraction (12). Alternatively, we can use equation (6) iteratively to calculate the same series of rational approximations – we set $x_0 = 1$, then substitute this value in equation (6) to find $x_1 = \frac{3}{2}$, and so on. Continuing in this manner we find the first few *convergents* of (12) to be

$$1 \quad \frac{3}{2} \quad \frac{7}{5} \quad \frac{17}{12} \quad \frac{41}{29} \quad \frac{99}{70} \quad \frac{239}{169} \quad \frac{577}{408} \quad \cdots$$

We can learn quite a bit about the convergence properties of regular continued fractions by examining these successive convergents a little more closely. Notice that

$$2 \times 1^2 = 1^2 + 1 \qquad 2 \times 2^2 = 3^2 - 1 \tag{13}$$

$$2 \times 5^2 = 7^2 + 1 \qquad 2 \times 12^2 = 17^2 - 1 \tag{14}$$

$$2 \times 29^2 = 41^2 + 1 \qquad 2 \times 70^2 = 99^2 - 1 \tag{15}$$

From these equalities we see that each successive convergent $\frac{y}{x}$ is a best possible rational approximation to $\sqrt{2}$ because each one comes within one unit of being exactly right.

$$2 \times x^2 = y^2 \pm 1 \tag{16}$$

We also see that the successive convergents alternate between being less than and greater than the limit they are approaching. In other words, they behave like the partial sums of a convergent series in which the sign alternates from term to term.

The Canonical Continued Fraction for an Arbitrary Real Number

Now that we have derived an example of a continued fraction, it's time to introduce some formal definitions. We define a *continued fraction* to be an expression in the form

$$a_0 + \cfrac{b_1}{a_1 + \cfrac{b_2}{a_2 + \cfrac{b_3}{a_3 + \ddots}}} \tag{17}$$

where the a_i and b_i are real or complex numbers. The b_i are the *partial numerators* of the continued fraction, and the a_i are the *partial denominators*. The numerical values obtained by truncating the continued fraction after a finite number of partial denominators are known as the *convergents* of the continued fraction – the first convergent is a_0, the second convergent is $a_0 + \frac{b_1}{a_1}$, and so forth.

If all the partial numerators b_i are equal to unity and all the partial denominators a_i after the *integer part* of the fraction, a_0, are natural numbers, the continued fraction is said to be a *regular (or canonical) continued fraction*. The general theory of continued fractions, especially those whose partial numerators and denominators are complex numbers, is quite long and involved. The remainder of this article deals exclusively with regular continued fractions.

Here is a simple algorithm to obtain the unique regular continued fraction representation of any real number. Let us suppose that we are given an arbitrary real number ω. We introduce the standard notation for the integer and fractional parts of ω – $[\omega]$ represents the greatest integer that is less than or equal to ω, and $\{\omega\}$ represents the remaining fractional part of ω, if any. If ω is an integer we set $a_0 = \omega$ and the algorithm is complete. If ω is not an integer, $0 < \{\omega\} < 1$, and we have

$$\omega = [\omega] + \{\omega\} = [\omega] + \frac{1}{\frac{1}{\{\omega\}}} \tag{18}$$

A pattern is beginning to emerge. We set $a_0 = [\omega]$ and $a_1 = \left[\frac{1}{\{\omega\}}\right]$, then proceed to evaluate the fractional part of $\frac{1}{\{\omega\}}$. If that fractional part is zero the algorithm is complete. Otherwise we repeat the process by setting

$$a_2 = \left[\frac{1}{\left\{\frac{1}{\{\omega\}}\right\}}\right] \tag{19}$$

and evaluate the remaining fractional part, if any. If ω is a rational number this process will terminate after a finite number of steps. If ω is irrational the remaining fractional part will never vanish, and the resulting regular continued fraction will be infinitely long.

Perhaps a simple example using a small rational number will make this process clearer. Let's expand the common fraction $\frac{83}{25}$ just to get some practice.

$$\begin{aligned} \frac{83}{25} &= 3 + \frac{1}{\frac{25}{8}} \\ &= 3 + \cfrac{1}{3 + \cfrac{1}{8}} \\ &= 3 + \cfrac{1}{3 + \cfrac{1}{7 + \cfrac{1}{1}}} \end{aligned} \tag{20}$$

There's a lot more one could say about continued fractions. Perhaps I'll return to write more about them another day. For now let's just summarize what we've learned:

– A continued fraction is an expression of the form

$$a_0 + \cfrac{b_1}{a_1 + \cfrac{b_2}{a_2 + \cfrac{b_3}{a_3 + \ddots}}} \tag{21}$$

where the a_i and the b_i are arbitrary real or complex numbers.

– If all the partial numerators b_i are equal to 1 and all the partial denominators a_i are integers, the continued fraction is a *regular* continued fraction.

– Every irrational number has a unique representation as an infinitely long regular continued fraction. Every rational number can be expressed as a finite regular continued fraction in two closely related ways: the tail end can be be written as either $\frac{1}{n}$, or as $\cfrac{1}{(n-1)+\frac{1}{1}}$.