More Trigonometry – The Law of Sines, etc.

A common problem in trigonometry is this: given some sides and angles, determine the unknown sides and angles of a planar triangle. Clearly if we are given all three sides of the triangle we have defined the triangle completely, but we would still like to know how to determine the three angles. If we are given all three angles, but none of the sides, there are infinitely many possibilities: we can make the triangle larger or smaller without changing any of the angles (these are *similar* triangles). If we are given one side and two of the three angles, one and only one triangle can fill the bill, because the third angle is just $180°$ less the sum of the other two angles. If we are given two sides and the angle between those sides, there is only one solution to the problem. When we are given two sides and an angle oppposite one of those sides, two solutions are possible, as we will now show.

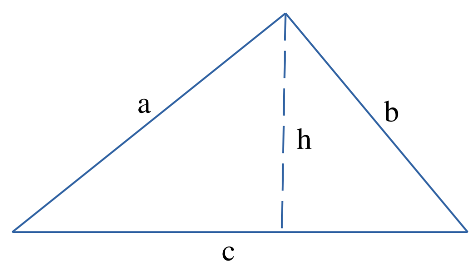

Figure 1: A Generic Triangle with Altitude h

In figure 1, above, the base of the triangle is side c, and we have drawn a dashed line h perpendicular to side c to delineate the height of the triangle. Angle A is understood to be opposite side a (angle at lower right), and angle B is opposite side b (lower left). Now by the definition of the sine function, and because the dotted line h divides Δabc into two right-angled triangles, we have two equations:

$$\sin A = \frac{h}{b} \quad \text{and} \quad \sin B = \frac{h}{a}$$
$$(\sin A) * b = h \quad \text{and} \quad (\sin B) * a = h$$
$$(\sin A) * b = (\sin B) * a = h \quad \Rightarrow \quad \frac{\sin A}{a} = \frac{\sin B}{b}$$

Since, in Figure 1 above, we might just as well have made side a the “base” of the triangle we see, by symmetry, that the *Law of Sines* is a triple equality:

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

We can use the law of sines directly to solve for a triangle when we are given two sides of a triangle and the angle opposite one of the given sides. The law of sines admits either zero, one, or two possible solutions in this case, as explained below.

The case where no solution is possible is probably the easiest one to understand. Suppose that we are given sides a and b, plus angle A, opposite side a. By the law of sines,

$$\frac{\sin A}{a} = \frac{\sin B}{b} \quad \Rightarrow \quad \sin B = \frac{\sin A * b}{a}$$

But $\sin B \leq 1$, for every angle B. So if side b is so long that $\frac{\sin A * b}{a} > 1$, there is no triangle Δabc to match the given criteria. Perhaps a simple example will make this clearer. Suppose $a = 10, b = 25$, and angle $A = 30°$. Since $\sin 30° = 1/2$, these criteria would have us satisfy $\sin B = (1/2) * 25/10 = 1.25$, and that is not possible.

The case where only one solution is possible is most easily understood as a special instance of the case with two solutions, so let's consider the dual solutions case next.

Given sides a and b, plus angle B, we can, in general, find two supplementary angles, A and $\pi/2 - A$, that both satisfy $\frac{\sin A}{a} = \frac{\sin B}{b}$.

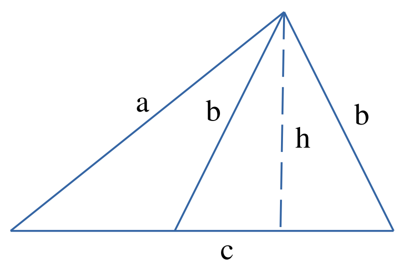

Figure 2: Acute and Oblique Triangles with Altitude h

In Figure 2, above, we see two triangles, one acute (all three angles $\leq 90°$) and one oblique (angle $A > 90°$). Sides a and b are the same in both triangles, and it should be clear that in the oblique triangle on the left, angle $A = 180°$, minus angle A in the acute triangle lying to the right (the auxiliary triangle with two sides of length b is isosceles, so two of the angles in it must be equal). Since $\sin\theta = \sin(\pi/2 - \theta)$ for $0 \leq \theta \leq \pi/2$, we can see why the law of sines sometimes admits two solutions, as illustrated above.

When is there only one solution? Well, if angle $A \leq$ angle B, $A - B > 0 \Rightarrow A + (\pi - B) > \pi$. But the sum of the angles in a triangle cannot exceed $\pi = 180°$. So if angle $A \leq$ angle B (that is, if $a \leq b$), and angle B plus sides a and b are given, the solution, if it exists, is unique.

Note that we can also use the law of sines to solve for a unique triangle when we are given one side and two of the three angles. For instance, given side a and angles B and C, we can set angle $A = 180° - B - C$, and then set $b = (a * \sin B)/\sin A$ and $c = (a * \sin C)/\sin A$.

The Law of Cosines: a Generalization of the Pythagorean Theorem

The Pythagorean Theorem, that in a right triangle the square of the hypotenuse equals the sum of the squares of the other two sides, is the most famous theorem in Euclidean geometry. Did you know that there is a generalized version of this theorem that applies to every planar triangle? It is called the *Law of Cosines*. The proof proceeds by dividing a generic triangle into two right triangles.

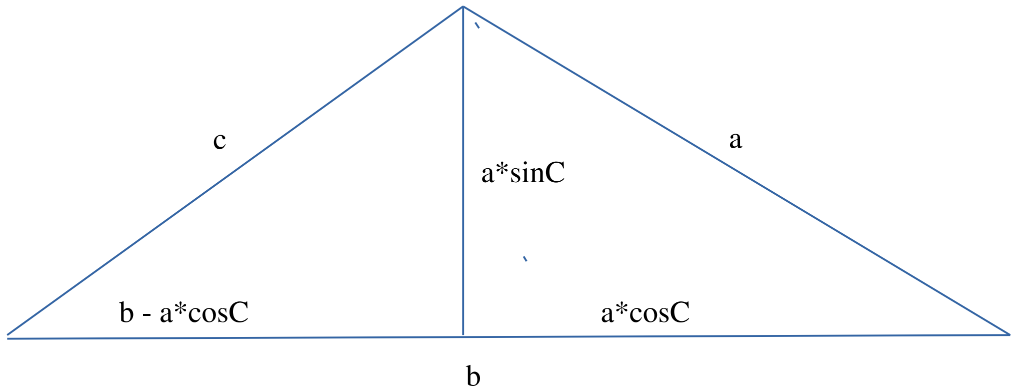

Figure 3: A Generic Triangle Split into Two Right Triangles

We will prove the law of cosines for Δabc by applying the Pythagorean Theorem to the right triangle on the left-hand side, defined by the vertical line segment in the middle of Figure 3. By the Pythagorean Theorem:

$$\begin{aligned} c^2 &= (a \sin C)^2 + (b - a \cos C)^2 \\ &= a^2 * \sin^2 C + b^2 - 2ab * \cos C + a^2 * \cos^2 C \\ &= a^2 * (\sin^2 C + \cos^2 C) + b^2 - 2ab * \cos C \\ &= a^2 + b^2 - 2ab \cos C \end{aligned}$$

Since the labels for the three sides of the triangle are arbitrary, we can write the law of cosines three different ways, one way for each of the three sides and its opposite angle.

$$\begin{aligned} a^2 &= b^2 + c^2 - 2bc \cos A \\ b^2 &= a^2 + c^2 - 2ac \cos B \\ c^2 &= a^2 + b^2 - 2ab \cos C \end{aligned}$$

We can also use the law of cosines to solve for the three angles A, B, and C when we are given the three sides a, b, and c:

$$\begin{aligned} \cos A &= (b^2 + c^2 - a^2)/2bc \\ \cos B &= (a^2 + c^2 - b^2)/2ac \\ \cos C &= (a^2 + b^2 - c^2)/2ab \end{aligned}$$

The three angles are then easily determined by applying the arccosine (aka $\cos^{-1}$) function: $A = \cos^{-1}(\cos A)$, etc. For instance, if we set $a = b = c = 3$ in the formulas above we obtain $\cos A = \cos B = \cos C = \frac{9}{18} = \frac{1}{2}$. And $\cos^{-1}\left(\frac{1}{2}\right) = \pi/3 = 60°$.