The Circular Functions: Introduction to Trigonometry

As its name suggests, *trigonometry* is all about measuring triangles: the three angles, and how the lengths of the three sides relate to those angles. Trigonometry is one of the oldest branches of mathematical analysis, having been studied by the ancient Sumerians almost 5,000 years ago. The early Greek mathematicians developed theorems about trigonometry using purely geometrical methods. Hipparchus, who flourished during the second century BC, produced and published the first tables of a trigonometric function. Trig has been around for a long time.

Traditionally, trigonometry has been studied in a flat, or Euclidian, space. If you have studied Euclidian geometry, you may remember that the sum of the three angles in any triangle is always $180°$. You probably also learned the Pythagorean Theorem: In any right-angled triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides. These two theorems lie at the heart of trigonometry. It is also possible to define the trigonemetric functions in a curved space, such as the surface of a sphere. We won't concern ourselves with such complications, at least, not yet.

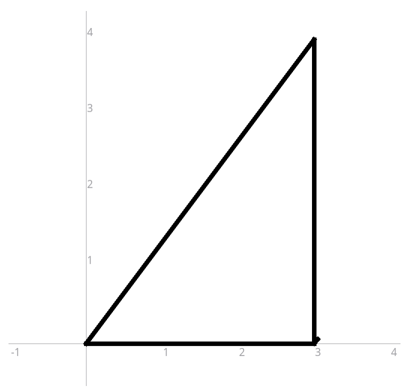

Figure 1: A Right-Angled Triangle

Before we plunge ahead, we should discuss the measurement of angles. The ancient Sumerians divided the circle into 360 equal parts, called degrees. So a right angle, or one-quarter of a circle, is an angle of $90°$, and all three internal angles of an equilateral triangle are $60°$. With the advent of the calculus some 350 years ago, mathematicians began measuring angles in radians, which use the length of the subtended arc of the unit circle to describe an angle. Since the entire circle has a length of $2\pi r$ $(= 2\pi$ when $r = 1)$, 2π radians are $360°$, π radians are $180°$, $\pi/2$ radians are $90°$, and so on.

This paper will primarily express angular measurements in radians, because that is a more natural way to express an angle, especially when it comes to things like the power series expansions of the trigonometric functions. Given a right-angled triangle with sides ABC, where θ is the angle between the adjacent side A and the hypotenuse C, and $\pi/2-\theta$ is the angle between the opposite side B and the hypotenuse C, we define the principal trigonometric functions this way:

$$\cos\theta = \frac{A}{C} \quad \sin\theta = \frac{B}{C} \quad \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{A}{B} \tag{1}$$

These three functions are read as “cosine theta”, “sine theta”, and “tangent theta”, respectively. (There are three reciprocal functions: secant, cosecant, and cotangent, that are also used from time to time. Those are defined by $\sec\theta = \frac{1}{\cos\theta}$, $\csc\theta = \frac{1}{\sin\theta}$, and $\cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}$.)

It is customary to present the triangle ABC as being inscribed inside a circle of radius 1, so that the length of the hypotenuse C is always 1. When presented this way, $\cos\theta$ is just the x-coordinate of the point on the circle subtended by the angle θ, and $\sin\theta$ is the y-coordinate of the same point. Since the equation of the unit circle centered at the origin is just $x^2 + y^2 = 1$, the fundamental trigonemetric identity says the same thing: $\sin^2\theta + \cos^2\theta = 1$. Here's a picture.

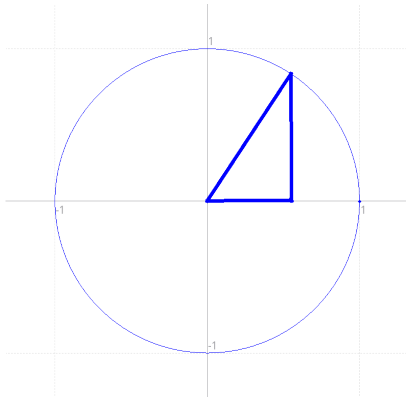

Figure 2: A Right Triangle Inscribed in the Unit Circle

According to the conventional rule, the values of the trigonometric functions are generated by traversing the unit circle counterclockwise: from the point $(1, 0) \quad (\theta = 0)$ to the point $(0, 1) \quad (\theta = \pi/2)$, thence through the point $(-1, 0) \quad (\theta = \pi)$, on through $(0, -1) \quad (\theta = 3\pi/2)$, and thence back to the starting point. The three basic trigonometric functions are said to be *periodic*, with period 2π:

$$\sin\theta = \sin(\theta \pm 2n\pi) \quad \text{and} \quad \cos\theta = \cos(\theta \pm 2n\pi)$$

for every natural number n. Here's a picture.

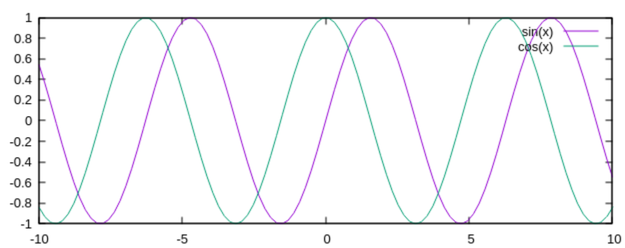

Figure 3: The periodic functions $\sin x$ and $\cos x$

For real arguments x, the domain of $\sin x$ and $\cos x$ is clearly $(-\infty, +\infty)$, and the range is $[-1, 1]$. These two functions are continuous throughout their domain. The tangent function $\tan x$ has a discontinuity whenever $\cos x = 0$; the reciprocal functions also have discontinuities at regular intervals ($\sec x$ at $\pi/2 \pm 2n\pi$; $\csc x$ and $\cot x$ at $0 \pm 2n\pi$).

The Inverse Functions Arccos, Arcsin, and Arctan

The trigonometric functions are periodic, therefore they do not have well-defined inverses. So we must adopt a convention about the "principal value" of each one of the six inverse functions. The table below shows the names of the six inverse trigonometric functions, two different ways they are represented symbolically, and the ranges in which their principal values fall.

Name	Symbolic	Alternative	Principal Value
Arccosine	$\arccos(x)$	$\cos^{-1} x$	$[0, \pi]$
Arcsine	$\arcsin(x)$	$\sin^{-1} x$	$[-\frac{\pi}{2}, \frac{\pi}{2}]$
Arctangent	$\arctan(x)$	$\tan^{-1} x$	$(-\frac{\pi}{2}, \frac{\pi}{2})$
Arcsecant	$\mathrm{arcsec}(x)$	$\sec^{-1} x$	$[0, \frac{\pi}{2})$
Arccosecant	$\mathrm{arccsc}(x)$	$\csc^{-1} x$	$(0, \frac{\pi}{2}]$
Arccotangent	$\mathrm{arccot}(x)$	$\cot^{-1} x$	$(0, \pi)$

The inverse functions do not play a large role in elementary trigonometry. They are more frequently encountered in calculus, where they crop up here and there as antiderivatives, aka indefinite integrals.

Formulas for the Sum and Difference of Two Angles

There are several ways to derive the formulas for the sine and cosine of the sum (or difference) of two angles α and β. The simplest way is to appeal to Euler's formula: $e^{ix} = \cos x + i \sin x$.

$$\begin{aligned}
\cos(\alpha+\beta) + i\sin(\alpha+\beta) &= e^{i(\alpha+\beta)} \\
&= e^{i\alpha} * e^{i\beta} \\
&= (\cos\alpha + i\sin\alpha) * (\cos\beta + i\sin\beta) \\
&= (\cos\alpha * \cos\beta - \sin\alpha * \sin\beta) + i(\sin\alpha * \cos\beta + \sin\beta * \cos\alpha)
\end{aligned}$$

or, equating real and imaginary parts in this equality,

$$\begin{aligned}
\cos(\alpha+\beta) &= (\cos\alpha * \cos\beta - \sin\alpha * \sin\beta) \\
\sin(\alpha+\beta) &= (\sin\alpha * \cos\beta + \sin\beta * \cos\alpha)
\end{aligned}$$

The formulas for the difference of angles fall out of these when we recognize the fact that $\sin(-\beta) = -\sin\beta$:

$$\begin{aligned}
\cos(\alpha-\beta) &= (\cos\alpha * \cos\beta + \sin\alpha * \sin\beta) \\
\sin(\alpha-\beta) &= (\sin\alpha * \cos\beta - \sin\beta * \cos\alpha)
\end{aligned}$$

Using these formulas, the fact that two of the angles in an isosceles triangle are $45° = \pi/4$ radians, and the fact that the side of a regular hexagon inscribed inside a circle is the same length as the radius of the circle, we can construct a simple table of sines and cosines as follows.

θ, rads	θ, degs	$\sin\theta$	$\sin$ (decimal)	$\cos\theta$	$\cos$ (decimal)
0	$0°$	0	.000000	1	1.000000
$\pi/12$	$15°$	$(\sqrt{6}-\sqrt{2})/4$	.258819	$(\sqrt{6}+\sqrt{2})/4$	.965926
$\pi/6$	$30°$	$1/2$	.500000	$\sqrt{3}/2$	.866025
$\pi/4$	$45°$	$\sqrt{2}/2$	.707107	$\sqrt{2}/2$	.707107
$\pi/3$	$60°$	$\sqrt{3}/2$	.866025	$1/2$	.500000
$5\pi/12$	$75°$	$(\sqrt{6}+\sqrt{2})/4$	.965926	$(\sqrt{6}-\sqrt{2})/4$	.258819
$\pi/2$	$90°$	1	1.000000	0	.000000

Early editions of trigonometric tables were derived starting from these rudimentary values, along with the half-angle trick, which I'll explain next. Let's begin by deriving the double-angle formulas, by setting $\alpha = \beta$ in the sum formulas on page 3, above.

$$\begin{aligned}
\cos(\alpha+\alpha) &= \cos 2\alpha \\
&= \cos^2\alpha - \sin^2\alpha \\
&= 1 - 2\sin^2\alpha \\
&= 2\cos^2\alpha - 1 \\
\sin(\alpha+\alpha) &= \sin 2\alpha \\
&= 2\sin\alpha\cos\alpha
\end{aligned}$$

where, in the formula for $\cos 2\alpha$ we have employed the fundamental identity $cos^2\alpha + \sin^2\alpha = 1$ two different ways, by adding, first, $\sin^2\alpha - \sin^2\alpha = 0$, and, second, $\cos^2\alpha - \cos^2\alpha = 0$ to the right-hand side, then grouping terms. Now we can derive the half-angle formulas by substituting $\frac{1}{2}\alpha$ for α in the preceding equations for $\cos 2\alpha$:

$$\cos\alpha = 1 - 2\sin^2\frac{\alpha}{2} \quad \Rightarrow \quad \sin\frac{\alpha}{2} = \sqrt{\frac{1-\cos\alpha}{2}}$$

$$\cos\alpha = 2\cos^2\frac{\alpha}{2} - 1 \quad \Rightarrow \quad \cos\frac{\alpha}{2} = \sqrt{\frac{1+\cos\alpha}{2}}$$

For many years, astronomers and surveyors used these half-angle formulas, together with the sum and difference formulas, to generate ever more detailed tables of the sine and cosine functions. They got a lot of practice computing square roots (since, in many cases, they resorted to $\cos\theta = \sqrt{1-\sin^2\theta}$ once they had a value for $\sin\theta$). Eventually, in 1715, the English mathematician Brook Taylor published his eponymous theorem, which simplified the calculation of tables of sines and cosines by introducing *power series expansions* for the trigonometric functions. We'll explore that topic in more depth later, when we learn about the differential calculus.