# Sequences and Series: Convergence and Divergence

Sequences of numbers, and the sum of a given sequence, play a large role in mathematical analysis. Let's begin our investigation with some definitions. A *sequence* is just an ordered set of real or complex numbers $\{a_i\}$. The order in which the numbers appear is important; the sequence $\{1, 2, 3, 4, \ldots\}$ is not the same sequence as $\{1, 3, 2, 4, \ldots\}$, even though those two sets contain the same elements. Sequences of real numbers $a_i$ are said to be increasing if, and only if, $a_i < a_{i+1}$ for every natural number $i$. A sequence of real numbers $b_i$ is decreasing if, and only if, $b_i > b_{i+1}$ for every natural number $i$. Sequences of complex numbers cannot be called increasing or decreasing, because there is no natural ordering of the complex numbers. But we can associate a real "magnitude" with each complex number (i.e., its distance from 0), so we can identify sequences of increasing (moving away from 0) or decreasing (moving towards 0) magnitude when we speak of complex numbers. A sequence may contain either a finite or an infinite number of items.

A *series* is formed by adding up some or all of the numbers in a sequence. If the sequence contains only finitely many elements, adding all of them up is just a matter of routine. Things get a lot more interesting when we attempt to add up all the numbers in a sequence containing an infinite number of items. To do that we must proceed carefully. We employ the customary $\Sigma$ notation for summation and, given a sequence $a_i$ we form a new sequence of *partial sums* $\{S_n\}$:

$$S_n = a_0 + a_1 + a_2 + \ldots + a_n = \sum_{i=0}^{n} a_i$$

If the new sequence $\{S_n\}$ approaches some unique limiting value $S$ as $n$ grows without bound, we say that $\lim n \to \infty(S_n) = S$, and we define the sum of all the infinitely many $a_i$ to be that limiting value.

$$S = \sum_{i=0}^{\infty} a_i$$

This may be more easily understood if we work with a concrete example. Let's consider the general *geometric sequence* with ratio $r$, given by $\{1, r, r^2, r^3, \ldots\}$,for which $a_i = r^i$. Let's think about the partial sums of this sequence and do some algebra.

$$S_n = 1 + r + r^2 + r^3 + \ldots + r^n$$
$$r * S_n = r + r^2 + r^3 + r^4 + \ldots + r^{n+1}$$

Now let's subtract the second equation from the first one; all the terms $r$, $r^2$, $r^3$, …, $r^n$ cancel out, leaving us with

$$(1 - r) * S_n = 1 - r^{n+1} \tag{1}$$
$$S_n = \frac{1 - r^{n+1}}{1 - r} \tag{2}$$

Now equation (1) is true for any ratio $r$, as we have just shown. For example, if $r = 2$, and $n = 3$, (1) says that $S_3 = \frac{1-2^4}{1-2} = \frac{-15}{-1} = 15$. And sure enough, $1+2+2^2+2^3 = 1+2+4+8 = 15$.

Now when $r \geq 1$ the partial sums of the geometric series will clearly grow without bound ... even $1+1+1+1+1+1+\dots$ will get very large if we throw in more and more 1s. The situation is different when $-1 < r < 1$; in that case the partial sums will always converge to a limit.

$$S_n = \frac{1-r^{n+1}}{1-r} = \frac{1}{1-r} - \frac{r^{n+1}}{1-r}$$

$$S = \lim_{n\to\infty} (S_n) = \frac{1}{1-r}$$

because $r^{n+1} \to 0$ when $-1 < r < 1$ and $n \to \infty$.

## An Example of a Divergent Series

Will a series of partial sums $S_n$ necessarily converge to a limit $S$ if the underlying sequence $\{a_i\}$ converges to zero? Not necessarily. Consider the harmonic sequence, $\{1,\ 1/2,\ 1/3,\ 1/4\dots\}$. We can easily show that the sum of these numbers grows without bound as the number of terms included becomes arbitrarily large by grouping successive terms judiciously:

$$\begin{aligned} S &= 1 + \left(\frac{1}{2}\right) + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \dots \\ &> 1 + \left(\frac{1}{2}\right) + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \dots \\ &= 1 + \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right) + \dots \end{aligned}$$

Obviously, $1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \dots$ grows without bound, and we can continue grouping $8$, then $16$, then $32$ terms, etc., as long as we like. So the sum of the harmonic sequence is unbounded; the harmonic series is divergent.

What happens if we start with the same sequence $\{1,\ \frac{1}{2},\ \frac{1}{3},\ \frac{1}{4},\dots\}$ and sum them up with alternating signs, like this: $S_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + - \dots$? Again, a judicious regrouping of terms gives us an answer. We write this sum two different ways.

$$\begin{aligned} S_{2n} &= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{5} - \frac{1}{6}\right) + \dots = \frac{1}{2} + \frac{1}{12} + \frac{1}{30} + \frac{1}{56} + \dots \\ S_{2n+1} &= 1 - \left(\frac{1}{2} - \frac{1}{3}\right) - \left(\frac{1}{4} - \frac{1}{5}\right) - \left(\frac{1}{6} - \frac{1}{7}\right) - \dots = 1 - \frac{1}{6} - \frac{1}{20} - \frac{1}{42} - \frac{1}{72} - \dots \end{aligned}$$

Clearly the partial sums $S_{2n}$ form an increasing sequence, and the partial sums $S_{2n+1}$ form a decreasing sequence. Looking a little more closely we see that $S_2 < S_3 > S_4 <$

$S_5 >< \ldots$. So the two sequences must approach the same limit, the $S_{2n}$ from below, and the $S_{2n+1}$ from above, each sequence being bounded by the other.

These results from the harmonic series suggest some definitions. If the sequence $\{|a_i|\}$ produces a convergent series of partial sums $S_n$, we say that the series $S$ is *absolutely convergent.* Notice that this definition applies to series of complex numbers, and to series of real numbers as well. If the signs of the numbers in a sequence alternate, or change in some other regular pattern, and the associated series converges, the series is said to be *conditionally convergent.* If a series converges absolutely, we can separate it into several different pieces, and all the pieces will still converge absolutely. We can't be so careless when a series is conditionally convergent; rearranging the terms in that case may cause the series to converge to a different limit, or not to converge at all.

## Two Simple Tests for Convergence

There are two different ways an infinite series can diverge: either by growing without bound, or by oscillation among two or more "limiting" values (an infinite series is convergent if, and only if, it approaches a *unique* limiting value). Series that grow without bound are basically all the same. Series that diverge by oscillation are more interesting, but not a great deal is known about them. In general, mathematicians are most interested in series that converge to a unique limit. So we are motivated to enunciate some practical tests that allow us to distinguish convergent series from divergent series.

We have already developed a mechanism to create the first of these tools, which is commonly known as the *comparison test.* This test works by comparing a new sequence $\{b_i\}$ to a known sequence $\{a_i\}$ term by term: if the $a_i$ are decreasing, and if the series of partial sums of the $a_i$ is known to converge, and if also, $b_i \leq a_i$ for every natural number $i$, then the series of partial sums of the $b_i$ will also converge:

$$a_{i+1} < a_i \quad \text{and} \quad b_i \leq a_i \quad \Rightarrow \quad \sum_{i=0}^{\infty} b_i \leq \sum_{i=0}^{\infty} a_i$$

In this situation the series formed by the $a_i$ are said to dominate the partial sums of the $b_1$. We can easily craft a divergence test from the same cloth: if the partial sums of the $a_i$ are known to diverge, and $b_i \geq a_i$, the partial sums of the $b_i$ will also diverge.

Here's a practical application of the comparison test. The exponential function $e^z$ is defined by an infinite power series:

$$e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!}$$

We can prove that this power series is absolutely convergent for every complex number $z$ by comparing it to the geometric series, which is known to converge whenever $|r| < 1$. For a given complex number $z_0$, there exists some natural number $n_0$ such that $|z_0| < n_0$ and, therefore, $|\frac{z_0}{n_0}| < 1$. But for indices $k > n_0$, the general term $z_0^k/k!$ is dominated by $|z_0^k|/n_0^k$ because $n_0+1 > n_0$. Hence the series $e^z$ is absolutely convergent for every complex number $z$.

Another common convergence test is the *alternating series* test, which we have already encountered in our discussion of the harmonic series, above. To recapitulate, briefly, if $\{|a_i|\}$ is decreasing, and if $\lim_{i\to\infty} a_i = 0$, the alternating series $a_0 - a_1 + a_2 - a_3 + - \ldots$ converges. We have already proved this assertion (see page 2, above).

## The Ratio Test and the Root Test

There are two more simple tests for convergence, both of which are similar to the comparison test.The *ratio test* asserts that, if the ratio $a_i/a_{i+1}$ tends to a limit as $i$ increases:

| | |
|---|---|
| The series converges when | $\lim_{i\to\infty}\left(\frac{\vert a_i\vert}{\vert a_{i+1}\vert}\right) < 1$ |
| The series diverges when | $\lim_{i\to\infty}\left(\frac{\vert a_i\vert}{\vert a_{i+1}\vert}\right) > 1$ |
| The test is inconclusive when | $\lim_{i\to\infty}\left(\frac{\vert a_i\vert}{\vert a_{i+1}\vert}\right) = 1$ |

The *root test* is similar; it asserts that when $|a_m|^{1/m}$ tends toward a limit:

| | |
|---|---|
| The series converges when | $\lim_{m\to\infty} \sqrt[m]{\vert a_m\vert} < 1$ |
| The series diverges when | $\lim_{m\to\infty} \sqrt[m]{\vert a_m\vert} > 1$ |
| The test is inconclusive when | $\lim_{m\to\infty} \sqrt[m]{\vert a_m\vert} = 1$ |

The proof of both assertions is easily proved by using a geometric series with an appropriate ratio $r$ in conjunction with the comparison test.

There is one more test I ought to mention here, because it is frequently useful when the other tests fail. It depends on the Fundamental Theorem of the Calculus, and it states that if the numbers in the sequence $a_n$ can be expressed as an integrable function, $f(n)$, then

$$\sum_{n=1}^{\infty} f(n) \quad \text{converges when} \quad \int_1^{\infty} f(x)dx \quad \text{is well defined.}$$

Explaining exactly what this means would take us too far afield; an explanation will have to wait until we have learned more about the integral calculus.