Introduction to Riemann Integration

About 2,200 years ago, in a city called Syracuse on the island of Sicily, a Greek mathematician named Archimedes calculated the value of π to roughly 3 decimal places by the *method of exhaustion*. He did this with purely geometrical methods. He started with a circle in which he inscribed a regular hexagon, and around which he circumscribed another regular hexagon. If the circle has a radius of 1, the total length of the six sides of the inscribed hexagon is six, because each side of the inscribed hexagon has the same length as the radius (the hexagon can be divided into six equilateral triangles). The sides of the circumscribed hexagon have a total length of $4 * \sqrt{3} = 6.928203\ldots$, because each side is of length $2/\sqrt{3} = 2\sqrt{3}/3$. Here's a picture.

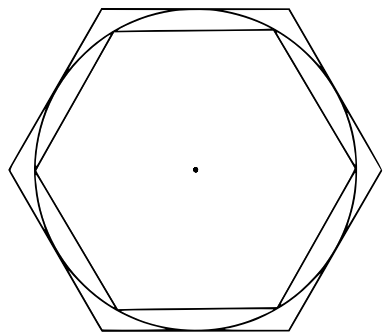

Figure 1: A Circle with Inscribed and Circumscribed Hexagons

Archimedes proceeded to draw the radii bisecting each side of the inscribed hexagon, then used the Pythagorean Theorem to calculate the length of the side of a regular dodecagon (12 equal sides) inscribed inside the same circle. (That length is $\sqrt{2-\sqrt{3}}$.) This led him to conclude that $3.1058\ldots < \pi$. He also drew the regular dodecagon circumscribing the circle, and proved that its side is of length $4 - 2\sqrt{3}$, implying that $\pi < 3.21539\ldots$. He continued in this fashion until he had polygons with 96 sides, yielding an amazingly accurate estimate of the value of π, given the crudity of arithmetic (i.e., Roman numerals) in Archimedes' day.

Today Archimedes' method of exhaustion is little more than a historical curiosity. Newton's integral calculus allows us to solve problems (like calculating the value of π) with more precision and with much less computational labor. But understanding Archimedes' reasoning helps us understand the limiting process that underlies *Riemann integration*, to which we now turn our attention.

Geometry teaches us how to calculate areas and volumes of polygonal figures formed with straight lines and flat faces. But what do we do when a planar figure, or a shape, is curved? Calculus tells us that we should approximate the curve with a sequence of step functions (i.e., rectangles that just fit under the curve), then refine the estimated area by making the rectangles narrower and narrower, and saying that, in the limit, as the width of the rectangles approaches zero, the area of the rectangles converges to the area under the curve. It's similar to Archimedes' method of exhaustion, but it uses rectangles instead of hexagons. This idea is illustrated below.

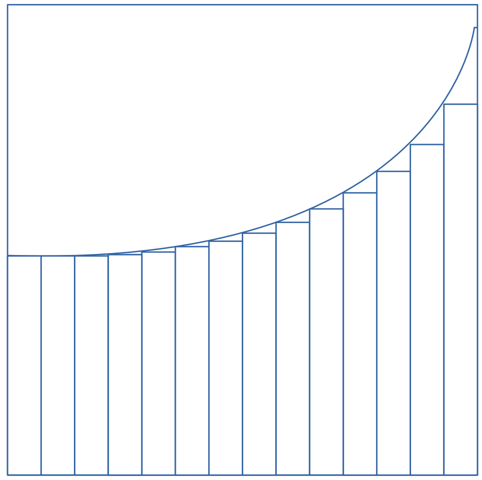

Figure 2: Underestimating the Area Under a Curve with Rectangles

The area beneath the curve in Figure 2 has been sub-divided into 14 rectangles. Since there are some gaps between the curve and the top sides of most of those rectangles, it's intuitively obvious that the area of the rectangles is less than the total area under the curve. What if we made each of the rectangles a bit taller, so the top right corner of each rectangle just touched the curve, and not the top left corner, as shown in Figure 2? Since all the space below the curve would now be filled in, and the left side of each modified rectangle would be above the curve, the area of the modified rectangles would be greater than the area below the curve. See Figure 3.

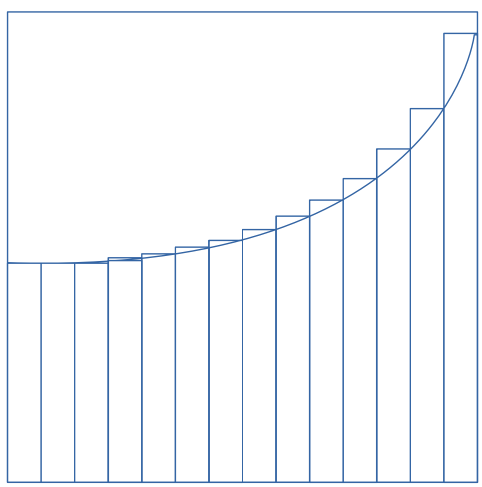

Figure 3: Overestimating the Area Under a Curve with Rectangles

Now that we have an intuitive understanding of Riemann integration, it is time for some definitions. We are given a real-valued function $f(x)$ defined everywhere on the closed interval $[a, b]$, $a < b$, some segment of the real number line. We define a *partition* of $[a, b]$ to be any finite set of n numbers $\{x_i\}$ such that $x_0 = a$, $x_n = b$, and $x_i < x_{i+1}$. We say that one partition P_1 is a *refinement* of another partition P_0 if, and only if, the set of points defining P_1 contains more elements than the set defining P_0. Conceptually, the intervals $[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]$ are the bases of the series of rectangles shown in Figures 2 and 3, above.

For each sub-interval $[x_i, x_{i+1}]$ defined by the partition P, we define m_i to be the minimum value $f(x)$ attains on that subinterval. Similarly, we define M_i to be the maximum value attained by $f(x)$ on the sub-interval $[x_i, x_{i+1}]$. Finally, we define two Riemann sums:

$$\text{lower Riemann sum} = \sum_{i=0}^{n-1} m_i * (x_{i+1} - x_i)$$
$$\text{upper Riemann sum} = \sum_{i=0}^{n-1} M_i * (x_{i+1} - x_i)$$

Now we stipulate that, if the limit of the lower Riemann sums as $n \to \infty$ exists, and if the upper Riemann sums converge to the same limit when $n \to \infty$, that limit is defined to be *the Riemann integral of* $f(x)$ *over the interval* $[a, b]$. In symbols:

$$\lim_{n\to\infty} \sum_{i=0}^{n-1} m_i * (x_{i+1} - x_i) = \int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=0}^{n-1} M_i * (x_{i+1} - x_i)$$

The *integral sign* $\int$ is a stylized letter s, much like the stylized double esses appearing in old documents like the Constitution, the Declaration of Independence, and the Magna Carta. It was first used by Wilhelm Gottfried Leibniz to denote a sum of “infinitesimal” quantities. Mathematicians are to some extent hide-bound by tradition: we still use Σ for a sum of discrete quantities, and $\int$ for a continuous sum of vanishingly small quantities. Although Newton and Leibniz invented the concept of an integral *ca.* 1650, it was Bernhard Riemann who first gave the rigorous definition of an integral presented above, in his doctoral dissertation (1854). That’s why we call it the Riemann integral, because Riemann was the first mathematician to define it rigorously.

Calulating the Integral $\int_0^A x^2 dx$ from First Principles

At this point it will be instructive to caculate the Riemann integral $\int_0^A x^2 dx$ from first principles, just to show that it can be done. This will entail quite a lot of work. Don’t worry. The *Fundamental Theorem of Calculus*, once we’ve proved it, will enable us to calculate the value of integrals like this one in a flash. Before we dive in, let’s prove a preliminary result we will need so we can complete the calculation of $\int_0^A x^2 dx$ from first principles.

$$\text{Lemma 1 :} \qquad \sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$$

The proof is by mathamatical induction. Clearly the formula is true when $n = 1$, because $(1 * 2 * 3)/6 = 1 = 1^2$. Now for the induction step. Suppose the formula is true for the sum of the first n perfect squares. We must show it holds when we add in $(n+1)^2$. We have

$$\begin{aligned}
\frac{n(n+1)(2n+1)}{6} + (n+1)^2 &= \frac{n+1}{6}(n)(2n+1) + (n+1)^2 \\
&= \frac{n+1}{6}((n)(2n+1) + 6(n+1)) \\
&= \frac{n+1}{6}(2n^2 + 7n + 6) \\
&= \frac{(n+1)(n+2)(2n+3)}{6} \\
&= \frac{(n+1)\left((n+1)+1\right)\left(2(n+1)+1\right)}{6}
\end{aligned}$$

and the induction is complete.

With Lemma 1 in hand, let us proceed to compute Riemann's lower sum for $f(x) = x^2$. We divide the interval $[0, A]$ into n equal pieces, each of width A/n, and observe that since $f(x) = x^2$ increases monotonically, the minimum value of $f(x)$ on the kth sub-interval is $\left(\frac{(k-1)A}{n}\right)^2$. So we can write the lower Riemann sum as

$$\begin{aligned}
\sum_{k=1}^{n} \left(\frac{(k-1)A}{n}\right)^2 \left(\frac{A}{n}\right) &= \frac{A^3}{n^3} \sum_{k=1}^{n} (k-1)^2 \\
&= \frac{A^3}{n^3} \left(\frac{(n-1)(n)(2n-1)}{6}\right) \\
&= A^3 \left(\frac{2n^3 - 3n^2 + n}{6n^3}\right)
\end{aligned}$$

from which it is clear that the lower Riemann sum $\to A^3/3$ as $n \to \infty$. And since the maximum value of $f(x)$ on the kth sub-interval is $\left(\frac{kA}{n}\right)^2$, we can write the upper Riemann sum as

$$\begin{aligned}
\sum_{k=1}^{n} \left(\frac{kA}{n}\right)^2 \left(\frac{A}{n}\right) &= \frac{A^3}{n^3} \sum_{k=1}^{n} k^2 \\
&= \frac{A^3}{n^3} \left(\frac{(n)(n+1)(2n+1)}{6}\right) \\
&= A^3 \left(\frac{2n^3 + 3n^2 + n}{6n^3}\right)
\end{aligned}$$

which also converges to $A^3/3$ as $n \to \infty$. So now we have proved that $\int_0^A x^2 dx = A^3/3$ by applying the definition of the Riemann integral directly.