How Henry Briggs Created Tables of Logarithms

I've been reading Henry Briggs' magnum opus *Arithmetica Logarithmica* (in English), and he goes into elaborate detail describing the process he used to flesh out his table of logarithms. Once he had a good value for $\ln(10)$, he might have taken a short cut and used some of Napiers' logarithms to get seven-place logs expressed in base 10. But he wanted 14-place logarithms, so he redid all of Napier's work from the ground up, using algorithms of his own devising.

Briggs recognized that, if one could compute the logarithms of enough prime numbers, one could fill in the logarithms of all the composite numbers that are less than an arbitrary limit, or at least get partway there, by simple addition and subtraction. For instance, $317^2 = 100,489$. There are $9,592$ prime numbers that are less than or equal to $100,000$; the other $90,497$ natural numbers from 2 through $100,000$ are composite. Each one of those composite numbers has at least one prime factor that is less than 317. And there are only 67 of those. So one can make a huge dent in the list of logarithms of natural numbers from 1 to $100,000$ by computing the natural logarithms of just 67 numbers the hard way. Those natural logarithms can be easily converted to base 10, usually by multiplication by $.434294481903252$, or by the equivalent long division (by 2.30258509299405).

The first prime number Briggs tackled was 2. Having wrestled with so many square root extractions already, he knew the task would be much easier if he could start with a number close to 1.000. Since $2^{10} = 1024$, he began by calculating $\sqrt{1.024}$. After extracting 47 successive square roots he reached $0.00000000000000016851605705394 98$; he multiplied this number by $1/\ln(10)$, doubled this product 47 times, then divided the sum by 10, obtaining $\log(2) \approxeq 0.301029995663981195$, good to 18 decimals. He subtracted this value from $1 = \log(10)$ to obtain $\log(5) \approxeq 0.698970004336018805$. So now he had the logarithms for $\{2, 4, 5, 8, 10, 16, 20, 25, 32, \ldots\}$; for numbers of the form $2^m * 5^n$, in other words.

The next prime number to be tackled was 3. Briggs knew that $6^9 = 10,077,696$. So he started by calculating $\sqrt{1.0077696}$. He proceeded as before, reaching $\ln(1.0077696)^{2^{-46}} \approxeq .000000000000000109985934588155718 66$ after extracting 46 successive square roots; converting this to a common logarithm and doubling that 46 times gave $\log(1.0077696) \approxeq .00336125345279269$; adding 7 and dividing by 9 gave $\log(6) \approxeq 0.77815125038364363$, and finally, subtracting $\log(2)$ yielded the result $\log(3) \approxeq 0.47712125471966244$ (17-digit accuracy).

Up to this point, Briggs had been extracting square roots by using the Binomial Theorem: if he had a trial value with say 3 digits, he had to square that number, subtract it from the first six digits of the number whose root was being extracted, append the next two digits of the number whose root was being extracted, then double the 3-digit result and use a trial and error method to find the 4th digit of the root he was extracting. This got to be very laborious when he was working with, say, 30 decimal places. Briggs kept meticulous notes, and eventually he noticed a pattern emerging, that looked something like this (these are the successive roots of 1.0077696).

1	00776	96000	00000	00000	00000
1	00387	72833	36962	45663	84655
1	00193	67661	36946	61675	87023
1	00096	79146	39099	01728	89072
1	00048	38402	68846	62985	49253

Look closely at the sequence of figures in the second column above. 776, 387, 193, 96, 48 are very nearly a geometric series with ratio $1/2$. Briggs formed additional differences between the differences, and differences betwen those, and eventually arrived at an approximation formula for the square root of numbers close to 1.00 that was equivalent to the Binomial Theorem expansion $\sqrt{1+x} = 1+(1/2)x-(1/8)x^2+(1/16)x^3-(5/128)x^4 \pm \ldots$, although it was expressed in a different form. From this point onward Briggs largely abandoned the old-school algorithm for extracting square roots in favor of this new method utilizing finite differences, because it converged to his "golden proportion" much more quickly than the older algorithm did.

Armed with this new difference method, Briggs started tackling more prime numbers. His method now involved taking the square root of a number close to 1.00 several times with the old Binomial Theorem algorithm and constructing a table of differences. When the differences were small enough, he started using his new algorithm, which converged pretty quickly. Then he worked backwards to get $\log(p)$. For instance, he observed that $7^2 = 49$, and that $48 = 3 * 2^4$; $50 = 2 * 5^2$. By simple algebra, $49^2 - 1 = 48 * 50$; that is, $2,401 - 1 = 2,400$. So he chose a starting value of $1 + (1/2400) = 1.00416666\ldots$, took the square root 44 times, then doubled the resulting logarithm 44 times and added $\log(2400) = \log(6) + \log(8) + \log(5) + \log(10)$, then divided the sum by 4, arriving at $\log(7) \approxeq 0.84509804001425682$, good to 16 decimal places (off by 1 in the 17th place).

Table 1: Finding Fractions Close to 1.000 for Small Primes

Prime	**Factors**	**Products**	**Dividend**	**Divisor**	**Quotient ...**
	$7*14$	98			
11	$9*11$	99	9801	9800	1.0001020408163265530...
	$10*10$	100			
	$7*50$	350			
13	$13*27$	351	123201	123200	1.0000081168831168831...
	$16*22$	352			
	$17*26$	442			
17	$21*21$	441	194481	194480	1.0000051419169066227...
	$11*40$	440			
	$17*19$	323			
19	$18*18$	324	104976	104975	1..0000095260776375327...
	$13*25$	325			
	$13*30$	390			
23	$23*17$	391	152801	152800	1.00000654450261780104...
	$28*14$	392			
	$23*24$	552			
29	$29*19$	551	303601	303600	1.00000329380764163372...
	$55*10$	550			

Table 1 above shows the figures Briggs used to get a rational fraction very close to 1.000 as the starting point to calculate the logarithms for the next few prime numbers after 7. He apparently used this method to calculate 14-place logarithms for all of the 21 2-digit primes. This allowed him to derive the logarithms for all the composite natural numbers up through $10,000$. Since he had explicitly calculated 14-place logarithms for the one- and two-digit primes, he was missing just 143 3-digit primes, and $1,061$ 4-digit prime numbers. So he had good values for roughly 88% of the first $100,000$ natural numbers.

Briggs' account of his method for calculating logarithms stops here; the rest of the *Arithmetica* is devoted to various methods of interpolation within the tables, calculation of antilogarithms, and a large variety of geometric problems that can be solved with logarithms. He published 14-place logarithms for the natural numbers from 1 to $20,000$ and from $90,000$ to $100,000$ in 1624. A few years later Mr. Adriaan Vlacq, a Dutchman, published a table of 10-place logarithms for all the natural numbers from 1 to $100,000$. About 1790 Georg Vega, from Ljubljana in present-day Slovenia, updated and improved Vlacq's tables, correcting many errors. Carl Friedrich Gauss apparently used Vega's tables to perform some of his astronomical calculations. Finally, between 1924 and 1952 Mr. Alexander John Thompson published *Logarithmetrica Britannica*, a new table of 20-place logarithms from 1 to $100,000$, in honor of the tercentenary of Briggs' *Logarithmica Arithmetica*.

The advent of electronic calculators and personal computers in the 1970s and '80s sounded a death knell for printed tables of logarithms. Why tote a huge book around when the handy little gadget in your pocket can churn out reams of accurate figures at a great rate of speed? In a sense, Briggs' tables destroyed their own utility, by making the solid-state electronic regime in which we live today an actuality. The engineers who created transistors and integrated circuits all depended on tables of logarithms to render their computations less laborious. I'm sure that Henry Briggs, were he alive today, would heartily approve of what later generations have accomplished with the aid of his cleverly constructed tables.