The Logarithm: a Forgotten Computational Tool

Years ago, long before the advent of electronic calculators and personal computers relieved people of the drudgery of performing numerical calculations by hand, every young math student had a table of five-place logarithms close at hand, and used it regularly. Today most young people haven't the foggiest notion of what a logarithm even *is*, much less how to use a table of logarithms to solve a problem in arithmetic. That's too bad. The logarithm function, variously denoted by $\ln(x)$, or $\log(x)$, plays an important role in higher mathematics, and understanding how logarithms work can prove very helpful to scientists and engineers when they're sketching out solutions to practical problems. Let's dive in!

So what is a logarithm, exactly? Briefly, it's a fractional exponent. Let's recall a couple of facts from elementary algebra. The *product rule* tells us that $b^x * b^y = b^{x+y}$ for any base $b > 0$ and any real numbers (and even for any complex numbers) x and y. Since division is the operation inverse to multiplication and subtraction is the inverse of addition, we may easily infer that, for a positive base b, $b^x/b^y = b^{(x-y)}$ for any real or complex numbers x and y. Similarly, the *power rule* tells us that $(b^x)^y = b^{(x*y)}$, which, by analogy with the product rule, also tells us that $b^{x/y} = (b^x)^{1/y}$ (which is a real number so long as $b > 0$).

Early in the 17th century, about 50 years before Isaac Newton discovered calculus, a Scottish mathematician named John Napier had a brilliant insight. We can devise a set of logarithms to some base $b > 0$ (and $\neq 1$), so that problems involving multiplication and division are simplified to a couple of table lookups, plus addition and subtraction. Similarly, problems involving exponentiation and root extraction can be solved using the table plus multiplication and division. All we have to do is define a *logarithm function* $\log_b(x)$ with the property that $x = b^{\log_b(x)}$, then compute $\log_b(x)$ for a wide range of values of x, somehow or another. The resulting table of logarithms can then be used to make a lot of arithmetic calculations a lot more easily than by using other manual methods.

Now constructing such a table was no easy task. Forget about that for a moment. Let's suppose that we had such a table, and we wanted to use it to calculate some number, say $\sqrt[5]{3}$. We would simply locate $\log_b(3)$ in the table, divide that number by 5, then read the table in reverse to get a value for the fifth root of 3. You can try it yourself by using the table of five-place logarithms that commences on page 4: $\log_{10}(3) = .47712$; $.47712/5 = .095424$; $\log_{10}(1.24) = .09432$; $\log_{10}(1.25) = .09691$. So $1.24 < \sqrt[5]{3} < 1.25$. A rough interpolation gives $\sqrt[5]{3} = 1.245$. And indeed, $(1.245)^5 = 2.991$, which is within 0.3% of 3. Not exactly right, but very close.

So how did John Napier construct his logarithmic tables? The notes he left us are fairly obscure, but it appears that he started with the number 1.0000001, then proceeded to create both arithmetic and geometric progressions using this quantity (for the GP) and this quantity less 1 (for the AP). For example, by squaring 1.0000001 22 times we arrive at $(1.0000001)^{4,194,304} = 1.52109486160268$; multiplying by $(1.0000001)^{(2^{21}+2^{19}+2^{16}+2^{15}+2^{14}+1040)}$ we estimate $\ln(2) = .6931472$, which is accurate to 7 decimal places when taking logarithms to the base e. So it appears that John Napier stumbled onto natural logarithms by accident; as later generations of mathematicians would prove, $\lim_{n\to\infty} = e$, and since $(1.0000001)^{10,000,000} = 2.71828169413208$, which is e to six decimal places, Napierian logarithms are often conflated with natural logarithms. This is not exactly true, but close enough for all practical purposes.

Table 1: Illustrating How Napier Probably Computed Logarithms

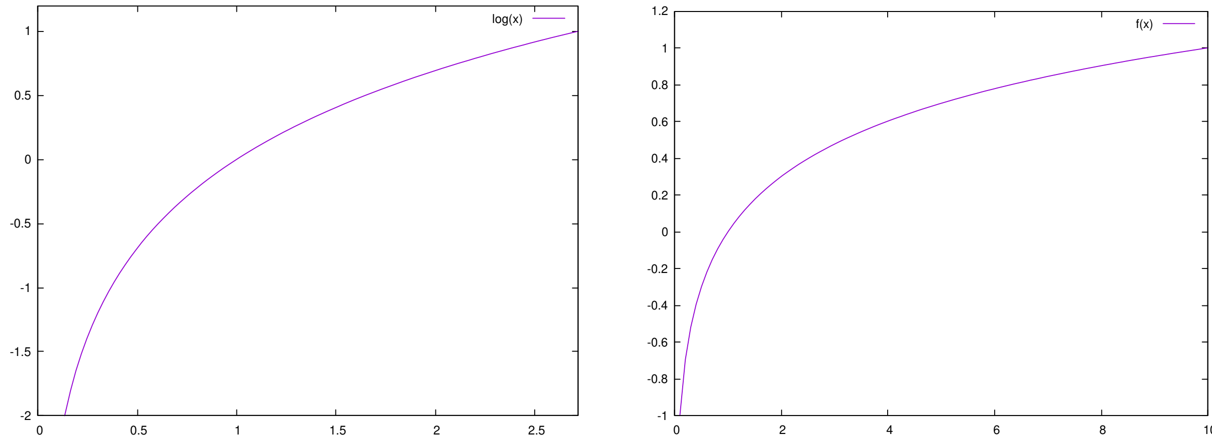
n	$10^{-7} * 2^n$	$(1.0000001)^{2^n}$
4	0.0000016	1.00000160000120
10	0.0001024	1.00010240523799
14	0.0016384	1.00163974282940
15	0.0032768	1.00328217441535
16	0.0065536	1.00657512149959
19	0.0524288	1.05382752415403
21	0.2097152	1.23332674567719
22	0.4194304	1.52109486160268
	Sum of column 2	Product of column 3
	0.6931472	1.99999997029744

Napier's logarithms were certainly useful, but hard to work with. An English mathematician named Henry Briggs read Napier's treatise in 1615, and collaborated with Napier on making the table of logarithms more readily useable. Briggs' big idea was to express logarithms to the base 10. Since many tradesmen and navigators were already conversant with decimal arithmetic, this innovation made logarithmic tables more easily understood by the common man.

Before we press ahead with a description of the procedure Briggs used to produce the very first tables of what we now call *common logarithms*, we should reflect a little while on the nature of the logarithmic function in general. In particular, what is the relationship between $\log_a(x)$ and $\log_b(x)$, where x is a fixed quantity other than 1? Let's do a little algebra.

$$\begin{aligned}
a^{\log_a(x)} &= x \\
\log_b(a^{\log_a(x)}) &= \log_b(x) \\
\log_a(x) * log_b(a) &= \log_b(x) \\
\log_a(x) &= \left(\frac{\log_b(x)}{\log_b(a)}\right)
\end{aligned}$$

where we have started with the definition of a logarithm to the base a, taken the log to the base b on both sides, applied the power rule for logarithms $\log_b(x^y) = y * \log_b(x)$, and finally divided both sides of the equation by the constant $\log_b(a)$. In other words, the only difference between logarithms taken to two different bases is a constant scaling factor $1/\log_b(a)$. A couple of pictures may make this clearer.



(a) Log to the base e (b) Log to the base 10

Figure 1: Except for a scaling factor, the shape of $\log_e(x)$ and $\log_{10}(x)$ is the same.

Now when John Napier set out to construct his table of logarithms, he only knew that he was trying to relate the numbers in a geometric progression to the numbers in an associated arithmetic progression. Henry Briggs had a much better idea of what he was trying to do. He understood the relationship between logarithms and exponents, and he also knew how to compute the square root of a number. So he started with the fact that $\log_{10}(\sqrt{10}) = 1/2$. Then clearly, $\log_{10}\left(\sqrt{\sqrt{10}}\right) = 1/4$, $\log_{10}\left(\sqrt{\sqrt{\sqrt{10}}}\right) = 1/8$, and so on.

Briggs made pretty good notes about the way he calculated his table of common logarithms. He carried 26 - 32 decimal places in his earliest calculations, eventually publishing tables with 14 decimal places. *Arithmetica Logarithmica*, published in London in 1624, contained 14-place common logarithms for the integers from 1 to $20,000$, and also from $90,000$ to $100,000$.

The first step in Briggs' process was to determine the natural logarithm of 10. He went about this very methodically, tabulating the successive values of $10^{2^{-n}}$ until $n = 54$. At this point the root extraction algorithm amounted to "Subtract 1, divide by 2, then add 1." In other words, the correction term x^2 in the identity $(1+x)^2 = 1 + 2x + x^2$ (on which the root extraction algorithm he was using is based) was negligible. He reasoned that the number he had extracted so laboriously – $1.00000000000000001278194932003235$, was 1 greater than $\ln(10)/2^{54}$. So he doubled $0.00000000000000001278194932003235$ 54 times, arriving at $\ln(10) \approxeq 2.3025850929940459145376326221824$, which is accurate to 15 decimal places. He had, in effect, found that $\ln(1+x) = 1 + x +$ (terms in $x^2, x^3, \ldots$ that can safely be ignored), simply by using the Binomial Theorem to extract square roots.

So now Briggs had the constant of proportionality between natural logarithms and logarithms to the base 10, but he still had to flesh out his tables. I'll leave a description of the way he did that for another day.

Table of Common Logarithms, aka Briggsian Logarithms

These are five-place logarithms to the base 10, for numbers from 1.00 to 9.99. For the range 10.0 to 99.9, add 1.00000 to tabulated values; for 100 to 999, add 2.00000, etc.

	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
.00	.00000	.30103	.47712	.60206	.69897	.77815	.84510	.90309	.95424
.01	.00432	.30320	.47857	.60314	.69984	.77887	.84572	.90363	.95472
.02	.00860	.30535	.48001	.60423	.70070	.77960	.84634	.90417	.95521
.03	.01284	.30750	.48144	.60531	.70157	.78032	.84696	.90472	.95569
.04	.01703	.30963	.48287	.60638	.70243	.78104	.84757	.90526	.95617
.05	.02119	.31175	.48430	.60746	.70329	.78176	.84819	.90580	.95665
.06	.02531	.31387	.48572	.60853	.70415	.78247	.84880	.90634	.95713
.07	.02938	.31597	.48714	.60959	.70501	.78319	.84942	.90687	.95761
.08	.03342	.31806	.48855	.61066	.70586	.78390	.85003	.90741	.95809
.09	.03743	.32015	.48996	.61172	.70672	.78462	.85065	.90795	.95856
.10	.04139	.32222	.49136	.61278	.70757	.78533	.85126	.90849	.95904
.11	.04532	.32428	.49276	.61384	.70842	.78604	.85187	.90902	.95952
.12	.04922	.32634	.49415	.61490	.70927	.78675	.85248	.90956	.95999
.13	.05308	.32838	.49554	.61595	.71012	.78746	.85309	.91009	.96047
.14	.05690	.33041	.49693	.61700	.71096	.78817	.85370	.91062	.96095
.15	.06070	.33244	.49831	.61805	.71181	.78888	.85431	.91116	.96142
.16	.06446	.33445	.49969	.61909	.71265	.78958	.85491	.91169	.96190
.17	.06819	.33646	.50106	.62014	.71349	.79029	.85552	.91222	.96237
.18	.07188	.33846	.50243	.62118	.71433	.79099	.85612	.91275	.96284
.19	.07555	.34044	.50379	.62221	.71517	.79169	.85673	.91328	.96332
.20	.07918	.34242	.50515	.62325	.71600	.79239	.85733	.91381	.96379
.21	.08279	.34439	.50651	.62428	.71684	.79309	.85794	.91434	.96426
.22	.08636	.34635	.50786	.62531	.71767	.79379	.85854	.91487	.96473
.23	.08991	.34830	.50920	.62634	.71850	.79449	.85914	.91540	.96520
.24	.09342	.35025	.51055	.62737	.71933	.79518	.85974	.91593	.96567
.25	.09691	.35218	.51188	.62839	.72016	.79588	.86034	.91645	.96614
.26	.10037	.35411	.51322	.62941	.72099	.79657	.86094	.91698	.96661
.27	.10380	.35603	.51455	.63043	.72181	.79727	.86153	.91751	.96708
.28	.10721	.35793	.51587	.63144	.72263	.79796	.86213	.91803	.96755
.29	.11059	.35984	.51720	.63246	.72346	.79865	.86273	.91855	.96802
.30	.11394	.36173	.51851	.63347	.72428	.79934	.86332	.91908	.96848
.31	.11727	.36361	.51983	.63448	.72509	.80003	.86392	.91960	.96895
.32	.12057	.36549	.52114	.63548	.72591	.80072	.86451	.92012	.96942
.33	.12385	.36736	.52244	.63649	.72673	.80140	.86510	.92065	.96988
.34	.12710	.36922	.52375	.63749	.72754	.80209	.86570	.92117	.97035
.35	.13033	.37107	.52504	.63849	.72835	.80277	.86629	.92169	.97081
.36	.13354	.37291	.52634	.63949	.72916	.80346	.86688	.92221	.97128
.37	.13672	.37475	.52763	.64048	.72997	.80414	.86747	.92273	.97174
.38	.13988	.37658	.52892	.64147	.73078	.80482	.86806	.92324	.97220
.39	.14301	.37840	.53020	.64246	.73159	.80550	.86864	.92376	.97267

Table of Common Logarithms, aka Briggsian Logarithms (cont.)

These are five-place logarithms to the base 10, for numbers from 1.00 to 9.99. For the range 10.0 to 99.9, add 1.00000 to tabulated values; for 100 to 999, add 2.00000, etc.

	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
.40	.14613	.38021	.53148	.64345	.73239	.80618	.86923	.92428	.97313
.41	.14922	.38202	.53275	.64444	.73320	.80686	.86982	.92480	.97359
.42	.15229	.38382	.53403	.64542	.73400	.80754	.87040	.92531	.97405
.43	.15534	.38561	.53529	.64640	.73480	.80821	.87099	.92583	.97451
.44	.15836	.38739	.53656	.64738	.73560	.80889	.87157	.92634	.97497
.45	.16137	.38917	.53782	.64836	.73640	.80956	.87216	.92686	.97543
.46	.16435	.39094	.53908	.64933	.73719	.81023	.87274	.92737	.97589
.47	.16732	.39270	.54033	.65031	.73799	.81090	.87332	.92788	.97635
.48	.17026	.39445	.54158	.65128	.73878	.81158	.87390	.92840	.97681
.49	.17319	.39620	.54283	.65225	.73957	.81224	.87448	.92891	.97727
.50	.17609	.39794	.54407	.65321	.74036	.81291	.87506	.92942	.97772
.51	.17898	.39967	.54531	.65418	.74115	.81358	.87564	.92993	.97818
.52	.18184	.40140	.54654	.65514	.74194	.81425	.87622	.93044	.97864
.53	.18469	.40312	.54777	.65610	.74273	.81491	.87679	.93095	.97909
.54	.18752	.40483	.54900	.65706	.74351	.81558	.87737	.93146	.97955
.55	.19033	.40654	.55023	.65801	.74429	.81624	.87795	.93197	.98000
.56	.19312	.40824	.55145	.65896	.74507	.81690	.87852	.96247	.98046
.57	.19590	.40993	.55267	.65992	.74586	.81757	.87910	.93298	.98091
.58	.19866	.41162	.55388	.66087	.74663	.81823	.87967	.93349	.98137
.59	.20140	.41330	.55509	.66181	.74741	.81889	.88024	.93399	.98182
.60	.20412	.41497	.55630	.66276	.74819	.81954	.88081	.93450	.98227
.61	.20683	.41664	.55751	.66370	.74896	.82020	.88138	.93500	.98272
.62	.20952	.41830	.55871	.66464	.74974	.82086	.88195	.93551	.98318
.63	.21219	.41996	.55991	.66558	.75051	.82151	.88252	.93601	.98363
.64	.21484	.42160	.56110	.66652	.75128	.82217	.88309	.93651	.98408
.65	.21748	.42325	.56229	.66745	.75205	.82282	.88366	.93702	.98453
.66	.22011	.42488	.56348	.66839	.75282	.82347	.88423	.93752	.98498
.67	.22272	.42651	.56467	.66932	.75358	.82413	.88480	.93802	.98543
.68	.22531	.42813	.56585	.67025	.75435	.82478	.88536	.93852	.98588
.69	.22789	.42975	.56703	.67117	.75511	.82543	.88593	.93902	.98632
.70	.23045	.43136	.56820	.67210	.75587	.82607	.88649	.93952	.98677
.71	.23300	.43297	.56937	.67302	.75664	.82672	.88705	.94002	.98722
.72	.23553	.43457	.57054	.67394	.75740	.82737	.88762	.94052	.98767
.73	.23805	.43616	.57171	.67486	.75815	.82802	.88818	.94101	.98811
.74	.24055	.43775	.57287	.67578	.75891	.82866	.88874	.94151	.98856
.75	.24304	.43933	.57403	.67669	.75967	.82930	.88930	.94201	.98900
.76	.24551	.44091	.57519	.67761	.76042	.82995	.88986	.94250	.98945
.77	.24797	.44248	.57634	.67852	.76118	.83059	.89042	.94300	.98989
.78	.25042	.44404	.57749	.67943	.76193	.83123	.89098	.94349	.99034
.79	.25285	.44560	.57864	.68034	.76268	.83187	.89154	.94399	.99078

Table of Common Logarithms, aka Briggsian Logarithms (cont.)

These are five-place logarithms to the base 10, for numbers from 1.00 to 9.99. For the range 10.0 to 99.9, add 1.00000 to tabulated values; for 100 to 999, add 2.00000, etc.

	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
.80	.25527	.44716	.57978	.68124	.76343	.83251	.89209	.94448	.99123
.81	.25768	.44871	.58092	.68215	.76418	.83315	.89265	.94498	.99167
.82	.26007	.44025	.58206	.68305	.76492	.83378	.89321	.94547	.99211
.83	.26245	.45179	.58320	.68395	.76567	.83442	.89376	.94596	.99255
.84	.26482	.45332	.58433	.68485	.76641	.83506	.89432	.94645	.99300
.85	.26707	.45484	.58546	.68574	.76716	.83569	.89487	.94694	.99344
.86	.26951	.45637	.58659	.68664	.76790	.83632	.89542	.94743	.99388
.87	.27184	.45788	.55771	.68753	.76864	.83696	.89597	.94792	.99432
.88	.27416	.45939	.58883	.68842	.76938	.83759	.89653	.94841	.99476
.89	.27646	.46090	.58995	.68931	.77012	.83822	.89708	.94890	.99520
.90	.27875	.46240	.59106	.69020	.77085	.83885	.89763	.94939	.99564
.91	.28103	.46389	.59218	.69108	.77159	.83948	.89818	.94988	.99607
.92	.28330	.46538	.59329	.69197	.77232	.84011	.89873	.95036	.99651
.93	.28556	.46687	.59439	.69285	.77305	.84073	.89927	.95085	.99695
.94	.28780	.46835	.59550	.69373	.77379	.84136	.89982	.95134	.99739
.95	.29003	.46982	.59660	.69461	.77452	.84198	.90037	.95182	.99782
.96	.29226	.47129	.59770	.69548	.77525	.84261	.90091	.95231	.99826
.97	.29447	.47276	.59879	.69636	.77597	.84323	.90146	.95279	.99870
.98	.29667	.47422	.59988	.69723	.77670	.84386	.90200	.95328	.99913
.99	.29885	.47567	.60097	.69810	.77743	.84448	.90255	.95376	.99957

A Usage Note: Toward the higher end of the table's range, the tabulated values lie, very nearly, on a straight line. For example, commencing with $\log(9.89)$ in the rightmost column above, the first differences are $.00044$, $.00043$, $.00044$, $.00044$, $.00044$, $.00043$, $.00044$, $.00044$, $.00043$, $.00044$. So people who were skilled in the use of logarithmic tables often used linear interpolation to estimate the logarithms of numbers that were not explicitly listed in the table. This approach adds about one additional digit of accuracy to the raw results for numbers whose first digit is greater than 2.