Understanding the Exponential Function $\exp(x)$

Now that we understand how to find the derivative of a polynomial of any degree, let's define a function that has the property $f'(x) = f(x)$ for every real number x. Let us also specify that $f(0) = 1$. Can we find a power series that defines such a function? Yes, we can. And it's surprisingly easy to do.

We begin with the constant term of the series. This term must be 1, because all the terms involving x^k vanish when $x = 0$. So we must have

$$f(x) = 1 + a_1x + a_2x^2 + a_3x^3 + \ldots$$

What is the coefficient a_1? Well, let's take the derivative of our series with unknown coefficients and equate it to the series for the function itself. (Remember, the derivative of $x^n = nx^{n-1}$.)

$$f'(x) = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \ldots = f(x) = 1 + a_1x + a_2x^2 + a_3x^3 + \ldots$$

So $a_1 = 1$. And since $f'(x) = f(x) \Rightarrow f'' = f(x)$, we can iterate this process.

$$\begin{aligned}
f'(x) &= a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \ldots \\
f''(x) &= (2*1)a_2 + (3*2)a_3x + (4*3)a_4x^2 + (5*4)a_5x^3 \ldots \\
f'''(x) &= (3*2*1)a_3 + (4*3*2)a_4x + (5*4*3)a_5x^2 + (6*5*4)x_6x^3 + \ldots \\
f''''(x) &= (4*3*2*1)a_4 + (5*4*3*2)a_5x + (6*5*4*3)x_6x^2 + (7*6*5*4)a_7x^3 + \ldots
\end{aligned}$$

from which we easily deduce that $a_2 = \frac{1}{2*1}$, $a_3 = \frac{1}{3*2*1}$, and, in general, $a_n = \frac{1}{n!}$. So we have found our power series:

$$f(x) = \exp(x) = 1 + \sum_{k=1}^{\infty} \frac{x^k}{k!} = \sum_{k=0}^{\infty} \frac{x^k}{k!} \tag{1}$$

where, in the last expression for our power series we adopt the conventional definitions $0^0 = 1$ and $0! = 1$, just to make the shorthand expression applicable to every x. Showing that this power series converges to a finite limit is fairly simple: we just use the comparison test for the tail-end of the series. Given any x, no matter how large, there is an integer k_x such that $k_x > x$. For all the terms in the series after the k_xth term, $x^{k_x+n}/(k_x+n)! < (x^{k_x})/(k_x)! * (x/k_x)^n$; in other words, the exponential series is dominated by a geometric series with ratio $x/k_x < 1$, and that geometric series is known to converge. So the series for e^x is absolutely convergent for every real number x (and for every complex number z, by the same argument).

There are several ways to show that there is a real number e such that $\exp(x) = e^x$, and that that number is given by $e = \exp(1)$. A brute force approach is used to prove this theorem in the next section.

Clearly the domain of $\exp(x)$ is $(-\infty, +\infty)$. Its range is $(0, +\infty)$. Here's a picture of $\exp(x)$.

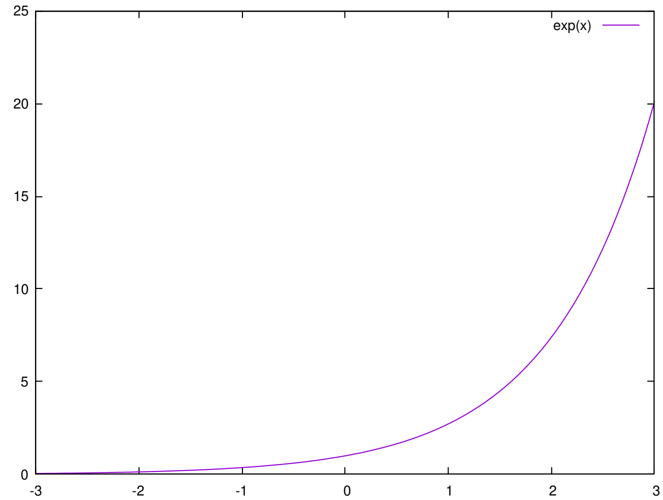

Figure 1: The Exponential Function $\exp(x)$

A Proof that $\exp(x) = e^x$

So I assert that there exists a real number $e = \exp(1)$, such that $\exp(x) = e^x$ for every real number x, where e^x represents the everyday exponentiation operation we know and love. To prove this we must show that the regular laws of exponents apply to $\exp(x)$, i.e., $\exp(x+y) = \exp(x) * \exp(y)$, $\exp(x)^n = \exp(nx)$, and $\exp(-x) = 1/\exp(x)$ for every pair of real numbers $\{x, y\}$ and for every real number n. Let us begin the proof by showing that $\exp(x)^n = \exp(nx)$ for every real number x and every natural number n. This first step is by mathematical induction. We will start by showing that $\exp(x)^2 = \exp(2x)$. The procedure is straightforward; since the infinite series for $\exp(x)$ is absolutely convergent we can multiply the series by itself term by term, as shown below.

$$
\begin{array}{cccccccccccccc}
1 & 1 & + & x & + & x^2/2! & + & x^3/3! & + & x^4/4! & + & x^5/5! & + & \dots \\
+ & & & & & & & & & & & & & \\
x & & & x & + & 2x^2/2! & + & 3x^3/3! & + & 4x^4/4! & + & 5x^5/5! & + & \dots \\
+ & & & & & & & & & & & & & \\
x^2/2! & & & & & x^2/2! & + & 3x^3/3! & + & 6x^4/4! & + & 10x^5/5! & + & \dots \\
+ & & & & & & & & & & & & & \\
x^3/3! & & & & & & & x^3/3! & + & 4x^4/4! & + & 10x^5/5! & + & \dots \\
+ & & & & & & & & & & & & & \\
x^4/4! & & & & & & & & & x^4/4! & + & 5x^5/5! & + & \dots \\
+ & & & & & & & & & & & & & \\
x^5/5! & & & & & & & & & & & x^5/5! & + & \dots \\
\vdots & & & & & & & & & & & & & \\
= & 1 & + & 2x & + & 4x^2/2! & + & 8x^3/3! & + & 16x^4/4! & + & 32x^5/5! & + & \dots \\
= & 1 & + & 2x & + & (2x)^2/2! & + & (2x)^3/3! & + & (2x)^4/4! & + & (2x)^5/5! & + & \dots \\
= & \exp(2x) & & & & & & & & & & & &
\end{array}
$$

Notice the binomial coefficients from Pascal's Triangle emerging in the columns of terms of the nth degree in the preceding table: we have used the fact that the sum of the numbers in the nth row of Pascal's Triangle is always 2^n to obtain the equalities at the end of the cross-multiplication of the two series. We will also use the binomial coefficients in the induction step, which follows. Suppose we have already proved that $\exp(x)^n = \exp(nx)$. We must show that $\exp(nx) * \exp(x) = \exp((n+1)x)$.

$$
\begin{array}{cccccccccccc}
1 & 1 & + & nx & + & (nx)^2/2! & + & (nx)^3/3! & + & (nx)^4/4! & + & \dots \\
+ & & & & & & & & & & & \\
x & & & x & + & 2nx^2/2! & + & 3n^2x^3/3! & + & 4n^3x^4/4! & + & \dots \\
+ & & & & & & & & & & & \\
x^2/2! & & & & & x^2/2! & + & 3nx^3/3! & + & 6n^2x^4/4! & + & \dots \\
+ & & & & & & & & & & & \\
x^3/3! & & & & & & & x^3/3! & + & 4nx^4/4! & + & \dots \\
+ & & & & & & & & & & & \\
x^4/4! & & & & & & & & & x^4/4! & + & \dots \\
+ & & & & & & & & & & & \\
\vdots & & & & & & & & & & & \\
= & 1 & + & (n+1)x & + & ((n+1)x)^2/2! & + & ((n+1)x)^3/3! & + & ((n+1)x)^4/4! & + & \dots \\
\vdots & & & & & & & & & & & \\
= & & & & & \exp((n+1)x) & & & & & &
\end{array}
$$

and the induction is complete. Clearly we have also shown that $\exp(x)^{\frac{1}{n}} = \exp(\frac{x}{n})$; simply substitute $y = x/n$ in the preceding argument and the rational root case is reduced to the integral power case (e.g., $\exp(x/2)^2 = \exp(x)$). So we have also $\exp(x)^{\frac{m}{n}} = \exp(\frac{m}{n}x)$. And since the rational numbers are a dense subset of the real numbers, the equality can be inferred for all positive real exponents.

Let us now turn our attention to negative exponents. We will show that $\exp(x) * \exp(-x) = 1$ for every real number x.

$$
\begin{array}{cccccccccccccc}
1 & 1 & + & x & + & x^2/2! & + & x^3/3! & + & x^4/4! & + & x^5/5! & + & \dots \\
- & & & & & & & & & & & & & \\
x & & - & x & - & 2x^2/2! & - & 3x^3/3! & - & 4x^4/4! & - & 5x^5/5! & - & \dots \\
+ & & & & & & & & & & & & & \\
x^2/2! & & & & & x^2/2! & + & 3x^3/3! & + & 6x^4/4! & + & 10x^5/5! & + & \dots \\
- & & & & & & & & & & & & & \\
x^3/3! & & & & & & - & x^3/3! & - & 4x^4/4! & - & 10x^5/5! & - & \dots \\
+ & & & & & & & & & & & & & \\
x^4/4! & & & & & & & & & x^4/4! & + & 5x^5/5! & + & \dots \\
- & & & & & & & & & & & & & \\
x^5/5! & & & & & & & & & & - & x^5/5! & + & \dots \\
\vdots & & & & & & & & & & & & & \\
= & 1 & + & 0 & + & 0 & + & 0 & + & 0 & + & 0 & + & \dots \\
= & 1 & & & & & & & & & & & &
\end{array}
$$

Again the binomial coefficients pop up, this time with alternating signs. And the binomial theorem with alternating signs says that $(1-1)^n = 0$.

There's still one more thing we need to prove before we conclude that $\exp(x) = e^x$. We need to show that the product rule for exponents obtains, that is, that $\exp(x) * \exp(y) = \exp(x+y)$. By now the procedure should be familiar.

$$\begin{array}{ccccccccccccc}
1 & 1 & + & x & + & x^2/2! & + & x^3/3! & + & x^4/4! & + & \ldots \\
+ & & & & & & & & & & & \\
y & & & y & + & 2xy/2! & + & 3x^2y/3! & + & 4x^3y/4! & + & \ldots \\
+ & & & & & & & & & & & \\
y^2/2! & & & & & y^2/2! & + & 3xy^2/3! & + & 6x^2y^2/4! & + & \ldots \\
+ & & & & & & & & & & & \\
y^3/3! & & & & & & & y^3/3! & + & 4xy^3/4! & + & \ldots \\
+ & & & & & & & & & & & \\
y^4/4! & & & & & & & & & y^4/4! & + & \ldots \\
\vdots & & & & & & & & & & & \\
= & 1 & + & (x+y) & + & (x+y)^2/2! & + & (x+y)^3/3! & + & (x+y)^4/4! & + & \ldots \\
= & \exp(x+y) & & & & & & & & & &
\end{array}$$

So now we know a little bit about the exponential function. It is greater than zero for every real number x. Its value, and the value of all its infinitely many derivatives, is the same at any point, its power series is given by $\exp(x) = \sum_{k=0}^{\infty}(x^k/k!)$, and it is also the real number e raised to the power x at every point x: $\exp(x) = e^x$.