Euler's Zeta Function: The Roots of Analytic Number Theory

Historians of mathematics credit Dirichlet (1805-1859) with writing the first paper applying the methods of analysis to problems in number theory. My own view is that Leonhard Euler (1707-1783) laid the foundations of analytic number theory with some of his early work on prime numbers and, in particular, by analyzing the real-valued zeta function, which Bernard Riemann would wield so imaginatively about a century later.

One of the most basic questions about prime numbers is "How many prime numbers are there?" Euclid (*ca* 300 BC) had used a simple proof by contradiction to show that there are an infinitude of primes. Euler produced an analytic proof of the same proposition. Now Euler loved to play with infinite series. One of his favorites was

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots = \sum_{k=0}^{\infty} x^k$$

Euler recognized that if he set $x = 1/p$ in this formula, where p is any prime number, he would have the equality

$$\left(1 - \frac{1}{p}\right) = 1 + \frac{1}{p} + \frac{1}{p^2} + \frac{1}{p^3} + \ldots$$

and, that if he multiplied all these sequences together somehow, the result would be the harmonic series:

$$\prod_{\text{Primes } p} \left(1 - \frac{1}{p}\right) = \sum_{n=1}^{\infty} \frac{1}{n}$$

by the fundamental theorem of arithmetic; multiplying all the powers of all the primes together gives us all the natural numbers. Since the harmonic series $\sum(1/n)$ is known to diverge, there must be infinitely many primes p.

This "proof", as Euler presented it, was not rigorous; one cannot use an "equality" like $\infty = \infty$ to "prove" a theorem (it amounts to dividing by zero). Later generations of mathematicians would patch this defect up by showing that, if $s > 1$, then both $\prod(1-1/p)^s$ and $\sum(1/n)^s$ are convergent, and they converge to the same limit, both of them diverging when $s = 1$. Euler's result then follows by letting $s \to 1^+$.

How Euler Evaluated $\zeta(2n)$

In 1734 Euler solved the "Basel problem", which had stumped the best mathematical minds in Europe for almost a century.Here's how he did it.

The Basel problem was easy to state. What is the value of $\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2}$? Euler quickly found the solution, utilizing his vast knowledge of power series for various functions. He began by considering the power series for $\sin x/x$:

$$\frac{\sin x}{x} = \left(\frac{1}{x}\right) * \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} \mp \ldots\right) = 1 - \frac{x^2}{3!} + \frac{x^4}{4!} \mp \ldots \tag{1}$$

Except for the fact that this power series has infinitely many terms, it looks just like a monic polynomial spelled backwards (that is, a polynomial whose leading coefficient is unity). So Euler conjectured that this power series for $\sin x/x$ can be expressed as an infinite product of factors $(1 - r_1) * (1 - r_2) * (1 - r_3) * \ldots$, where $\{r_1, r_2, r_3, \ldots\}$ is the set of roots of $\sin x/x$. He couldn't prove that the product was equal to the infinite series, but he relied on his mathematical intuition, which, as it turns out, was correct. (Weierstrass proved the eponymous Entire Function Factorization Theorem about 1850.) Now the function $\sin x/x$ vanishes if and only if $x = \pm n\pi$, where n is a natural number. So Euler was motivated to write this equation:

$$\begin{aligned}\left(\frac{\sin x}{x}\right) &= \left(1+\frac{x}{\pi}\right) * \left(1-\frac{x}{\pi}\right) * \left(1+\frac{x}{2\pi}\right) * \left(1-\frac{x}{2\pi}\right) * \left(1+\frac{x}{3\pi}\right) * \left(1-\frac{x}{3\pi}\right) * \ldots \\ &= \left(1-\frac{x^2}{\pi^2}\right) * \left(1-\frac{x^2}{(2\pi)^2}\right) * \left(1-\frac{x^2}{(3\pi)^2}\right) * \left(1-\frac{x^2}{(4\pi)^2}\right) * \ldots\end{aligned}$$

Evidently, the coefficient of x^2 in this infinite product is just

$$-\frac{1}{\pi^2} - \frac{1}{(2\pi)^2} - \frac{1}{(3\pi)^2} - \frac{1}{(4\pi)^2} - \ldots = \left(\frac{-1}{\pi^2}\right) \sum_{k=1}^{\infty} \frac{1}{k^2}$$

Equating this coefficient with the corresponding coefficient in (1) above we obtain

$$-\frac{1}{3!} = \left(\frac{-1}{\pi^2}\right) \sum_{k=1}^{\infty} \frac{1}{k^2} \quad \Rightarrow \quad \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6} \tag{2}$$

Euler didn't stop with $\zeta(2)$. At roughly the same time that he found $\zeta(2) = \pi^2/6$ he also published values for $\zeta(4), \zeta(6), \zeta(8), \zeta(10)$, and $\zeta(12)$. And within five years he had discovered a general formula for every even natural number,

$$\zeta(2n) = \frac{(-1)^{(n-1)}(2\pi)^{2n}}{2 * (2n)!} B_{2n}$$

where B_{2n} is the $2n$th Bernoulli Number. That Euler was one smart cookie.