Systems of Linear Equations & the Quadratic Formula

We have already learned how to "solve" the general linear equation with one unknown $a * x + b = c * x + d$. All we had to do was isolate the terms involving x on one side of the equation, then isolate the constant terms on the other side of the equation, and finally divide both sides by the coefficient of x. What happens when the equation has two unknowns?

Figure 1 below shows that a linear equation in two unknowns, in this case $y = x + 2$, defines an infinity of points lying along a line in the $x - y$ plane.

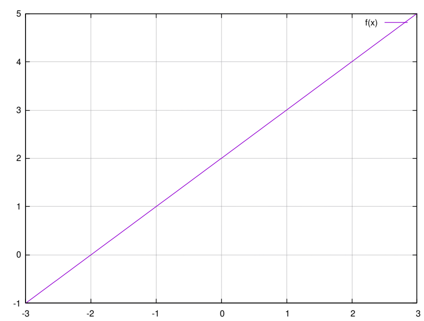

Figure 1: The equation $y = x + 2$

The graph of $y = x + 2$ is a straight line passing through the points $(-3, -1)$ and $(3, 5)$. When $x = 0$, $y = 2$. We call the point $(0, 2)$ the *y-intercept* of the equation $y = x + 2$, and we say that the line in Figure 1 has a *slope* of 1, because when x increases by one unit, y also increases by one unit: $\Delta x = \Delta y \Rightarrow \frac{\Delta x}{\Delta y} = 1$. In *analytic geometry* we say that the general equation of a line lying in the x-y plane is

$$y = mx + b$$

where m is the slope of the line and b is the y-intercept.

In general, if we have two equations in two unknowns, the two lines will cross; we call the point of intersection the "solution" for this particular system of equations.

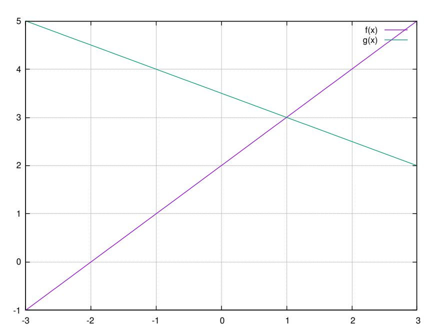

Figure 2: Two equations: $y = x + 2$ and $y = -.5x + 3.5$

There is one special case when a system of two linear equations in two unknowns will not have a "solution". If the two lines defined by the equations have the same slope, they will be parallel, and never intersect.

Do we have to draw a graph to solve a system of two linear equations in two unknowns? No, of course not. If the equations have been reduced to the form $y = mx + b$ the solution is simple.

$$\begin{aligned}
y &= x + 2 \\
y &= -.5x + 3.5 \\
x + 2 &= -.5x + 3.5 \\
1.5x &= 1.5 \\
x &= 1 \\
y &= 1 + 2 = 3
\end{aligned}$$

so the solution of our pair of equations is the ordered pair $(1, 3)$, or $x = 1$, $y = 3$. Here we have set two different expressions for y equal to each other, solved the resulting equation for x, then substituted the value $x = 1$ back into one of the original equations to find y.

If we can solve two equations in two unknowns, what about three equations in three unknowns? Well, if a single solution exists, we can find it this way.

$$\begin{aligned}
ax + by + cz &= j \\
dx + ey + fz &= k \\
gx + hy + iz &= l
\end{aligned}$$

We multiply the first equation by $f = f$ and the second one by $c = c$; then we multiply the original first one by $i = i$ and multiply the third one by $c = c$, and do some subtractions, ending with two equations in two unknowns.

$$\begin{aligned}
(af)x + (bf)y + (cf)z &= fj \\
(cd)x + (ce)y + (cf)z &= ck \\
(af - cd)x + (bf - ce)y &= (fj - ck) \\
(ai)x + (bi)y + (ci)z &= ij \\
(cg)x + (ch)y + (ci)z &= cl \\
(ai - cg)x + (bi - ch)y &= (ij - cl)
\end{aligned}$$

Now we have two equations in two unknowns; we use the same trick to eliminate y.

$$\begin{aligned}
(af - cd)x + (bf - ce)y &= (fj - ck) \\
(ai - cg)x + (bi - ch)y &= (ij - cl) \\
(af - cd)(bi - ch)x + (bf - ce)(bi - ch)y &= (fj - ck)(bi - ch) \\
(ai - cg)(bf - ce)x + (bf - ce)(bi - ch)y &= (bf - ce)(ij - cl) \\
((af - cd)(bi - ch) - (ai - cg)(bf - ce))x &= (fj - ck)(bi - ch) - (bf - ce)(ij - cl) \\
x &= \left(\frac{(fj - ck)(bi - ch) - (bf - ce)(ij - cl)}{(af - cd)(bi - ch) - (ai - cg)(bf - ce)} \right)
\end{aligned}$$

provided that $(af-cd)(bi-ch)-(ai-cg)(bf-ce) \neq 0$. Notice that we could have multiplied the second equation by $i = i$ and the third one by $f = f$ to eliminate the term in z. We could have eliminated x from the equations, then solved for either y or z. Or we could have permuted the three equations into any one of six different orders. Applying the same procedure will always produce the same solution x_0. We can find the full solution (x_0, y_0, z_0) by substituting the value x_0 into one of the equations with two unknowns, then substituting x_0 and y_0 into any one of the three original equations.

We could press ahead and look at systems of equations with four or five or even more unknowns. But doing that efficiently requires some new notation (matrices and vectors) associated with *vector spaces*. For now, let's move on to an analysis of the general quadratic equation $ax^2 + bx + c = 0$.

Solving the General Quadratic Equation

Solving the general linear equation $ax + b = cx + d$ was fairly straightforward. Solving the general quadratic equation. or equation of the second degree, requires a little more ingenuity. We shall assume that all the non-zero terms have been collected on the left-hand side of the equation:

$$ax^2 + bx + c = 0$$

We begin by subtracting $c = c$, then dividing by $a = a$.

$$\begin{aligned} ax^2 + bx &= -c \\ x^2 + \left(\frac{b}{a}\right) x &= -\left(\frac{c}{a}\right) \end{aligned}$$

Here comes the tricky part. Since $(x+k)^2 = x^2 + 2kx + k^2$, we are motivated to "complete the square" so we can remove the term in x from our equation. Setting $2kx = \frac{b}{a}x \Rightarrow k = \frac{b}{2a}$ we see that the left-hand side of the equation will become a perfect square if we just add $(\frac{b}{2a})^2 = (\frac{b}{2a})^2$ to our modified equation.

$$\begin{aligned} x^2 + \left(\frac{b}{a}\right) x + \left(\frac{b}{2a}\right)^2 &= \left(\frac{b}{2a}\right)^2 - \left(\frac{c}{a}\right) \\ \left(x + \frac{b}{2a}\right)^2 &= \left(\frac{b}{2a}\right)^2 - \left(\frac{c}{a}\right) \\ x + \frac{b}{2a} &= \pm\sqrt{\left(\frac{b}{2a}\right)^2 - \frac{4a}{4a}\left(\frac{c}{a}\right)} \\ x &= -\frac{b}{2a} \pm \sqrt{\frac{b^2 - 4ac}{4a^2}} \\ &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \end{aligned}$$

This last expression for x is known as the *quadratic formula*. The quantity $b^2 - 4ac$ is known as the *discriminant*. If it is positive, the original quadratic has two distinct real roots. If it is zero, the two roots are equal. And if the discriminant is negative, the roots are complex numbers.